SPECIALIZATIONS OF GENERALIZED DRINFEL'D-SOKOLOV HIERARCHIES

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ANDREW NORMAN HILL

University of Canterbury
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To my father, Norman Peter, and in memory of my mother,
Allison Margaret, for all they have done for me.

Abstract

The primary objective of this work is to extend the existing body of research on specializations of Drinfel'd-Sokolov hierarchies of partial differential equations and to provide some examples thereof.

The hitherto established results on specializations are reinterpreted and their intrinsic Lie algebraic essentials analysed. This permits the extension of these ideas to the so-called generalized Drinfel'd-Sokolov hierarchies developed in recent years.

The crucial generalization is that of a certain root space automorphism which is then lifted appropriately to the associated affine Lie algebra. Most importantly, this automorphism is shown to be gradation preserving, with respect to a gradation associated with the relevant Heisenberg subalgebra. Such an automorphism then lends itself to defining specific specializations of the generalized Drinfel'd-Sokolov hierarchies.

Integrable systems result from these constructions that are new in the sense that they are associated with nonstandard twisted type algebras determined by the fixed point subspace of the aforementioned automorphism.

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Chapter 1

Introduction

In the last two decades, much progress has been made in the field of non-linear partial differential equations and in particular as to what constitutes the notion of integrability. There have been attempts to formally prescribe methods of testing for integrability, perhaps most notably in [21]. So-called integrable equations have been observed to possess many of the properties of the celebrated Korteweg-de Vries (KdV) equation [46, 45]. These include the infinite dimensional prolongation algebras of Wahlquist and Estabrook [51] and the existence of infinitely many conservation laws and symmetries [47]. Many also possess the Painlevé property [53]. Also many admit a bi-Hamiltonian formulation [44, 47, 16]. However, most importantly for the purpose of this thesis, a great many are known to be expressible in zero-curvature form, a nomenclature which has its origins in the theory of principal bundles with connection. Earlier authors have explored this interplay [29, 12], though all that remains of it in the contemporary literature is the terminology.

It will be enough for the requirements of this work to note that a zero-curvature equation may be represented by a so-called Lax pair formulation [59]. This takes the form

$$-M_t + N_x + [M, N] = 0,$$

corresponding to

$$D\chi = d\chi + \tfrac{1}{2}[\chi, \chi] = 0,$$

where $D\chi$ is the Lie algebra-valued curvature 2-form of

$$\chi = M\ dx + N\ dt,$$

which is consequently referred to as a zero-curvature 1-form [34].

In 1981, and later more fully in 1985, Drinfel'd and Sokolov [17, 18] developed a theory of zero-curvature systems that expanded on the work of Zakharov and Shabat [59] and were able to show that there are hierarchies of partial differential equations associated to *any* affine Lie algebra. The latter were developed by Kač, Moody, Peterson and others [35, 36, 37].

The discoveries of Drinfel'd and Sokolov significantly enhanced the existing knowledge of zero-curvature systems. In particular, they provided an explanation of the intimate connection between the KdV equation and $\mathfrak{a}_1^{(1)}$, represented by the loop algebra of 2×2 trace free matrices. Furthermore, an analogue of the Miura transformation linking the modified KdV (mKdV) and KdV equations was uncovered. Here, the important algebraic feature is that the counterpart of the mKdV equation is associated with the principal gradation of the affine Lie algebra in question, whereas an analogue of the KdV equation is associated with the homogeneous gradation. If there exists any other minimal inequivalent gradation, then there is more than one analogue of the KdV equation for that particular affine Lie algebra.

A crucial element of these constructions was a maximal Abelian subalgebra possessing a homogeneously graded basis, known as the principal (respectively homogeneous) Heisenberg subalgebra in the case of the principal (respectively homogeneous) gradation. In 1992, de Groot *et al.* [13, 7, 30, 20, 4, 5, 6] were inspired by the idea of Wilson [55] to use gradations other than the principal and homogeneous gradations. The primary hurdle was in finding a suitable analogue of the principal and homogeneous Heisenberg subalgebras. To this end, they drew on a body of purely algebraic work, in particular that of Kač and Peterson [37], Dynkin [19] and Carter

[9], which established that the Heisenberg subalgebras of an affine Lie algebra are in one-to-one correspondence with the conjugacy classes of the Weyl group of the underlying finite dimensional Lie algebra. This enabled de Groot *et al.* to generalize the Drinfel'd-Sokolov (D-S) hierarchies, by replacing the principal or homogeneous Heisenberg subalgebra with an arbitrary Heisenberg subalgebra. Moreover, by introducing a partial ordering on the set of gradations of an affine Lie algebra, they were also able to generalize the concept of a Miura transformation, by showing that the hierarchy corresponding to a gradation "above" another in the partial ordering may be transformed to the hierarchy of the second "lower" gradation.

A further step was recently taken by Delduc and Fehér [14, 15] who constructed the elements of the different Heisenberg subalgebras necessary for the generalized Drinfel'd-Sokolov (GD-S) hierarchies. They also distinguished which of these were *regular*, a crucial feature in the material presented here.

In the earlier stages of the development of D-S hierarchies, Guil [25, 26, 27] addressed the issue of "specializing" D-S hierarchies. This notion had been earlier discussed in [41, 42, 55]. It consists of introducing relationships amongst the independent functions appearing in the partial differential equations of a D-S hierarchy in such a way as to produce a new consistent set of equations with a lesser number of independent functions. Until the work of Guil, this had only been done in a somewhat *ad hoc* and sporadic manner.

Guil's achievement was to provide an algebraic prescription for specializing the modified KdV hierarchies of the principal Heisenberg subalgebra of $\mathfrak{a}_n^{(1)}$. This was done by introducing a root space automorphism which lifted to an automorphism of the whole algebra. By restricting attention to the fixed point subspace of this automorphism, which in the principal case is itself a standard affine Lie algebra, Guil was able to construct specializations of any given hierarchy relating to the principal Heisenberg subalgebra of $\mathfrak{a}_n^{(1)}$. In doing so, the Calogero-Degasperis equation [8] was recovered and shown to be linked to $\mathfrak{a}_1^{(1)}$ as a "deformation" of the standard mKdV equation. Thus, these specialized hierarchies constitute yet another interesting class of integrable systems.

The work of this thesis has been to generalize Guil's method and apply it to the GD-S hierarchies of de Groot *et al.*. This has been done in the case of $\mathfrak{a}_n^{(1)}$ for those Heisenberg subalgebras corresponding to Weyl group elements which admit a regular eigenvector. The non-regular cases are still a largely unexplored area although Fordy [22] has made some progress in this direction.

The key to generalizing Guil's approach is to identify the intrinsic algebraic core of his work, so as to obviate his reliance on the special reflectional and rotational symmetries of the extended Dynkin diagram of $\mathfrak{a}_n^{(1)}$. This also suggests how the method might be applied to other affine Lie algebras.

In Chapter 2, the necessary facts about Lie algebras are presented, including the pivotal notions of root space and Dynkin diagram automorphisms and their lifts to the whole algebra (§2.3, §2.4). The material is based on Humphreys' book [33]. In §2.5, the concept of an affine Lie algebra and its gradations is developed, following the text by Helgason [28].

Chapter 3 explores the link between the conjugacy classes of the Weyl group and the collection of all possible Heisenberg subalgebras of any given affine Lie algebra. The idea of a Carter diagram is introduced in §3.1. The lifts of the conjugacy classes for $\mathfrak{a}_2$ and $\mathfrak{a}_3$ are listed by way of example in §3.2. In §3.3, a method for determining the gradation induced by the lift of a Weyl element is presented and in §3.4, the link with Heisenberg subalgebras is established. The different inequivalent Heisenberg subalgebras of $\mathfrak{a}_2^{(1)}$ and $\mathfrak{a}_3^{(1)}$ are given as examples.

Chapter 4 develops the theory of GD-S hierarchies. The essential background is outlined in §4.1. Some technicalities on the order of the lift of a Weyl group element are discussed in §4.2. In §4.3, the necessary theorems on so-called "gauge symmetries" are established.

In Chapter 5, the generalization of Guil's automorphism is presented. In §5.1, it is defined on the root space and lifted in §5.2 to the whole algebra. In §5.3, it is shown that this automorphism preserves the gradation associated with the corresponding

Weyl group element, a vital property in order to specialize GD-S hierarchies.

Finally, in Chapter 6, it is shown that the automorphism of Chapter 5 may be used to specialize GD-S hierarchies. The algebraic considerations are examined in §6.1 and some examples are presented in §6.2, with the aid of the symbolic algebra package MAPLE [10]. A few concluding remarks are contained in §6.3.

Appendix A presents an example of a slightly more general specialization than those considered in the main body of the thesis.

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Chapter 2

Algebraic Preliminaries

In this chapter, the necessary facts about Lie algebras are presented. In §2.1, some fundamental definitions are supplied, with mention of the standard representations of the classical simple Lie algebras. Only a brief mention is made of Lie groups in §2.2 in order to introduce the concept of inner automorphisms. In §2.3, Cartan subalgebras, roots and the Killing form are defined and in §2.4, Dynkin diagrams are introduced as is the lift of a root space automorphism to one of the whole algebra. The material in these sections follows Humphreys [33], though some of the notation is borrowed from Helgason [28]. Finally, in §2.5, Helgason's exposition of affine Lie algebras, their root spaces and gradations, from Chapter X §5 of [28], is presented and furnished with some examples of gradations on $\mathfrak{a}_2^{(1)}$ and $\mathfrak{a}_2^{(2)}$.

2.1 Basic Definitions Relating to Lie Algebras

To begin with we survey the background material on Lie algebras necessary for our purposes. The approach follows that in [33, 28].

A Lie algebra is a complex vector space $\mathfrak{g}$ endowed with a bilinear operation called the bracket, or commutator,

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

such that $(x, y) \mapsto [x, y]$, such that

$$[x, x] = 0 \ \forall x \in \mathfrak{g},$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \ \forall x, y, z \in \mathfrak{g} \quad \text{(Jacobi identity)}.$$

A Lie algebra isomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a vector space isomorphism such that

$$\varphi[x, y] = [\varphi x, \varphi y] \ \forall x, y \in \mathfrak{g}_1.$$

Standard examples are the matrix representations of $n \times n$ complex matrices where $[A, B] := AB - BA$. An algebra $\mathfrak{g}$ is said to be Abelian or commutative if $[\mathfrak{g}, \mathfrak{g}] = 0$.

Given any $x \in \mathfrak{g}$, there is associated a linear operator, ad $x : \mathfrak{g} \to \mathfrak{g}$, called the *adjoint* of x, defined by

$$\text{ad } x(y) = [x, y].$$

We shall be primarily concerned with the *simple* Lie algebras which are those Lie algebras $\mathfrak{g}$ possessing no nontrivial ideals (a subspace $\mathfrak{k} \subset \mathfrak{g}$ is said to be an ideal if $[\mathfrak{k}, \mathfrak{g}] \subset \mathfrak{k}$) and such that $[\mathfrak{g}, \mathfrak{g}] \neq 0$. The finite dimensional simple Lie algebras have been classified up to isomorphism. The so-called classical types are summarised as follows:

The algebras $\mathfrak{a}_n$, with dimension $n(n+2)$ and matrix representation $\mathfrak{sl}(n+1)$, given by the $(n+1) \times (n+1)$ matrices with trace zero.

The algebras $\mathfrak{b}_n$, with dimension $n(2n+1)$ and matrix representation $\mathfrak{o}(2n+1)$, given by the $(2n+1) \times (2n+1)$ matrices X which satisfy $SX = -X^T S$, where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix},$$

and I_n is the $n \times n$ identity matrix.

The algebras $\mathfrak{c}_n$, with dimension $n(2n+1)$ and matrix representation $\mathfrak{sp}(2n)$, given by the $2n \times 2n$ matrices X which satisfy $SX = -X^T S$, where

$$S = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The algebras $\mathfrak{d}_n$, with dimension $n(2n-1)$ and matrix representation $\mathfrak{o}(2n)$, given by the $2n \times 2n$ matrices X which satisfy $SX = -X^T S$, where

$$S = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

The remaining simple Lie algebras are given by the so-called exceptional Lie algebras $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$.

A Lie algebra is said to be *semisimple* if it may be written as a direct sum of simple ideals

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$$

so that $[\mathfrak{g}_i, \mathfrak{g}_j] = 0, i \neq j$.

Some more definitions are necessary for our requirements.

A Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if each of its elements is *ad-nilpotent*, by which is meant $\forall x \in \mathfrak{g} \ \exists \ k \in \mathbb{Z}_+$ such that $(\text{ad } x)^k = 0$.

An element is said to be ad-semisimple if ad x is diagonalizable or, equivalently, the roots of its minimal polynomial over $\mathbb{C}$ are distinct.

The *normalizer* of a subspace $\mathfrak{k} \subset \mathfrak{g}$ is defined by $N_{\mathfrak{g}}(\mathfrak{k}) = \{x \in \mathfrak{g} : [x, \mathfrak{k}] \subset \mathfrak{k}\}$.

The *centralizer* of a subset $A \subset \mathfrak{g}$ is defined by $\mathfrak{z}_{\mathfrak{g}}(A) = \{x \in \mathfrak{g} : [x, A] = 0\}$.

A criterion for determining semisimplicity of a Lie algebra involves the notion of the Killing form, a symmetric bilinear form $K : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, defined by

$$K(x, y) = \text{trace}(\text{ad } x \circ \text{ad } y).$$

K is associative in the sense that $K([x, y], z) = K(x, [y, z])$ and for any automorphism σ of $\mathfrak{g}$, $K(\sigma x, \sigma y) = K(x, y)$, as follows from the properties of the trace of a linear operator. The Lie algebra $\mathfrak{g}$ is semisimple if and only if K is nondegenerate.

2.2 Lie Groups and Inner Automorphisms

The simple Lie algebras may be interpreted as the set of left invariant vector fields on an associated Lie group, G, as outlined in [52]. This is a differentiable manifold with group structure such that the map $G \times G \to G$ given by $(g_1, g_2) \mapsto g_1 g_2^{-1}$ is C^∞. In the case of $\mathfrak{sl}(n+1)$, the associated Lie group is $SL(n+1)$, the $(n+1) \times (n+1)$ matrices of determinant 1. There is a well known map called the *exponential map*, denoted exp, from $\mathfrak{g} \to G$ such that

$$\begin{aligned} \exp(t_1 + t_2)x &= \exp(t_1 x)\exp(t_2 x), \\ \exp(-t_1 x) &= \exp(t_1 x)^{-1}, \end{aligned}$$

for all scalars t_1, t_2. For the classical algebras represented by square matrices, the exponential is just given by the usual exponentiation of matrices, which explains the choice of terminology.

We denote the automorphisms of $\mathfrak{g}$ by $\mathrm{Aut}(\mathfrak{g})$. An *endomorphism* of $\mathfrak{g}$ is defined to be any linear operator from the vector space $\mathfrak{g}$ into itself. We denote the set of all such linear operators over $\mathfrak{g}$ by $\mathrm{End}(\mathfrak{g})$, which itself becomes a Lie algebra on defining

$$[\varphi_1, \varphi_2] = \varphi_1 \circ \varphi_2 - \varphi_2 \circ \varphi_1, \ \ \forall \varphi_1, \varphi_2 \in \mathrm{End}(\mathfrak{g}).$$

It may be shown that $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathrm{End}(\mathfrak{g})$ and the exponential map is given by

$$\exp(\varphi) = \mathrm{id} + \varphi + \frac{1}{2!}\varphi^2 + \frac{1}{3!}\varphi^3 + \cdots$$

For the adjoint map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ given by

$$\mathrm{ad}\ x(y) = [x, y],$$

it may be shown that there exists an analogous map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ called the *adjoint representation*, such that $\mathrm{Ad} \circ \exp = \exp \circ \mathrm{ad}$. Under the matrix representation of $\mathfrak{g}$ and G,

$$\mathrm{Ad}\ B(y) = ByB^{-1}$$

where $B \in G, y \in \mathfrak{g}$. That is to say the group adjoint action is merely conjugation by B. Thus if $B = \exp(x)$ for some $x \in \mathfrak{g}$, then

$$\begin{aligned}\text{Ad } \exp(x)(y) &= \exp(\text{ad } x)(y)\\ &= y+[x,y]+\frac{1}{2!}[x,[x,y]]+\frac{1}{3!}[x,[x,[x,y]]]+\cdots\end{aligned}$$

The class of automorphisms of $\mathfrak{g}$ generated by automorphisms of the form

$$\text{Ad } \exp(x) = \exp(\text{ad } x),\ x \in \mathfrak{g}$$

is called the group of *inner automorphisms* of $\mathfrak{g}$ and is denoted by $\text{Int}(\mathfrak{g})$.

2.3 Cartan Subalgebras and Roots

For an arbitrary Lie algebra, a *Cartan subalgebra* (henceforth abbreviated to CSA) is a nilpotent self-normalizing subalgebra. In the semisimple (and therefore simple) case, this may be shown to be equivalent to requiring that the subalgebra be maximal with the property that each of its elements is ad-semisimple. Conforming to standard notation, we shall denote a CSA of $\mathfrak{g}$ by $\mathfrak{h}$. It may be shown that $\mathfrak{h}$ is maximal Abelian, and $\mathfrak{h} = \mathfrak{z}(\mathfrak{h})$. The standard example in matrix representations of the classical type Lie algebras is the subalgebra of diagonal matrices. Any two CSA's of a semisimple Lie algebra are conjugate under $\text{Int}(\mathfrak{g})$. The *rank* of a Lie algebra $\mathfrak{g}$ is defined to be the dimension of its CSA.

Let $\mathfrak{h}$ be a CSA of the semisimple Lie algebra $\mathfrak{g}$. For $\alpha \in \mathfrak{h}^*$, the dual space of $\mathfrak{h}$, we define

$$L^\alpha = \{x \in \mathfrak{g} : [h,x] = \alpha(h)x\ \forall h \in \mathfrak{h}\}.$$

If $L^\alpha \neq \{0\}$, then α is said to be a *root* (of $\mathfrak{g}$ with respect to $\mathfrak{h}$) with associated root space L^α. The Jacobi identity ensures that $[L^\alpha, L^\beta] \subset L^{\alpha+\beta}$. The root spaces L^α comprise the ad-nilpotent elements of $\mathfrak{g}$. If Δ denotes the set of nonzero roots, then $\mathfrak{g}$ admits the vector space direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} L^\alpha,$$

and each L^α, $\alpha \in \Delta$, is one dimensional. If α, β are roots such that $\alpha + \beta \neq 0$ then $K(L^\alpha, L^\beta) = 0$. The restriction of K to $\mathfrak{h} \times \mathfrak{h}$ is nondegenerate and for all $\alpha \in \mathfrak{h}^*$ there exists a unique $h_\alpha \in \mathfrak{h}$ such that $K(h, h_\alpha) = \alpha(h)$ for all $h \in \mathfrak{h}$. We define

$$\langle \lambda, \mu \rangle := K(h_\lambda, h_\mu).$$

If $\alpha \in \Delta$, then $-\alpha \in \Delta$ and $[L^\alpha, L^{-\alpha}] = \mathbb{C}h_\alpha$, where $\alpha(h_\alpha) = K(h_\alpha, h_\alpha) \neq 0$.

Furthermore, if $\mathfrak{h}_{\mathbb{R}} := \sum_{\alpha \in \Delta} \mathbb{R} h_\alpha$, in other words the *real* span of the h_α, then K is real and strictly positive definite so that K is an inner product over the Euclidean space $\mathfrak{h}_{\mathbb{R}}$.

A subset Π of Δ is called a *base* if Π is a basis for $\mathfrak{h}_\alpha$ and each root β may be expressed as

$$\beta = \pm \sum k_j \alpha_j, \;\; k_j \in \mathbb{Z}_+, \alpha_j \in \Pi.$$

The elements α_j of Π are called the *simple* roots. A root β is called *positive* if $\beta = +\sum k_j \alpha_j$ and *negative* if $\beta = -\sum k_j \alpha_j$. We write $\beta \succ 0$ or $\beta \prec 0$ respectively. The number of simple roots is equal to the rank of the algebra.

Associated to any simple Lie algebra of rank n is the $n \times n$ Cartan matrix whose (i,j)-entry a_{ij} is defined to be

$$a_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}.$$

In the case of $\mathfrak{a}_n$, $a_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}$. The Lie algebra of rank n is said to have *canonical generators* x_j, y_j, h_j, $1 \leq j \leq n$, where

$$h_j := \frac{2}{\langle \alpha_j, \alpha_j \rangle} h_{\alpha_j},$$

and $x_j \in L^{\alpha_j}$, $y_j \in L^{-\alpha_j}$, such that

$$[x_j, y_j] = h_j.$$

In the standard matrix representation of $\mathfrak{a}_n$, namely $\mathfrak{sl}(n+1)$,

$$h_j = E_{jj} - E_{j+1,j+1}, \;\; x_j = E_{j,j+1}, \;\; y_j = E_{j+1,j},$$

where $0 \leq j \leq n$ and the subscripts are taken modulo $n+1$, where E_{jk} is the matrix with 1 in the (j,k)-entry and zero everywhere else.

The root space corresponding to the simple root α_j is spanned by x_j. In the case of $\mathfrak{a}_n$, each root α is given by

$$\alpha = \pm \sum_{k=l_1}^{l_2} \alpha_k,$$

where $1 \leq l_1 \leq l_2 \leq n$. The corresponding root space is spanned by

$$E_{l_1,l_2+1} = [x_{l_1}, \ldots, x_{l_2}] := \text{ad } x_{l_1} \circ \cdots \circ \text{ad } x_{l_2-1}(x_{l_2}),$$

if $\alpha \succ 0$, and

$$E_{l_2+1,l_1} = [y_{l_2}, \ldots, y_{l_1}] := \text{ad } y_{l_2} \circ \cdots \circ \text{ad } y_{l_1+1}(y_{l_1}),$$

if $\alpha \prec 0$.

2.4 Dynkin Diagrams and their Symmetries

Each simple Lie algebra is uniquely determined by its root system, which, in turn, is characterised by its Cartan matrix. The Cartan matrix may be conveniently visualised by its *Dynkin diagram.* For an algebra of rank n, the Dynkin diagram is the graph of n vertices, the jth joined to the kth $(j \neq k)$ by $a_{jk}a_{kj}$ edges. It may be shown that $a_{jk}a_{kj}$ only ever takes the values $0, 1, 2$ or 3. The jth vertex may be thought of as representing the simple root α_j. If all the simple roots are of the same length, as is the case for $\mathfrak{a}_n$, $\mathfrak{d}_n$ and the exceptional algebras, the description of the Dynkin diagram is complete. If, however, this is not the case, it turns out that there are exactly two different simple root lengths, and on joining the vertices corresponding to roots of different length, assuming $a_{jk}a_{kj} \neq 0$, an arrow is added pointing to the shorter of the two roots. This happens for $\mathfrak{b}_n$ and $\mathfrak{c}_n$.

The Dynkin diagram of $\mathfrak{a}_n$ is shown in Figure 2.1.

As a graph, a Dynkin diagram may possess nontrivial graph automorphisms. These are permutations of the vertices preserving the multiplicity of the links and

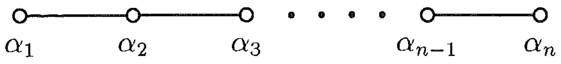

Figure 2.1: Dynkin diagram for $\mathfrak{a}_n$

the arrows. Such an automorphism turns out to have order $1, 2$ or 3, the instances of order 2 occurring only for $\mathfrak{a}_n$ for all $n > 1$ and $\mathfrak{d}_n$ for $n > 4$; the only Dynkin diagram possessing an automorphism of order 3 is $\mathfrak{d}_4$. In particular, for $\mathfrak{a}_n$, there is only one nontrivial diagram automorphism which we shall denote by ν, which sends the ith vertex to the $(n+1-i)$th vertex, thus representing a reflection through a vertical axis through the midpoint of the diagram.

Finally, we describe how root space automorphisms may be lifted to produce an automorphism on the whole algebra $\mathfrak{g}$. This is the content of §14.2 of [33] and plays a pivotal role in this thesis. First, a root space automorphism is a vector space isomorphism φ, such that

$$\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \frac{\langle \varphi(\beta), \varphi(\alpha) \rangle}{\langle \varphi(\alpha), \varphi(\alpha) \rangle}.$$

Note that φ need not be an isometry. The significance of the quantity $\langle \beta, \alpha \rangle / \langle \alpha, \alpha \rangle$ will become apparent when we introduce the Weyl group. However, for $\mathfrak{a}_n$ it turns out that φ is always an isometry as φ maps bases to bases and the simple roots of any base always have the same length.

Given a set of simple roots Π, of the root space, Δ, and a root space automorphism φ, then for $x_\alpha \in L^\alpha$, fix an arbitrary $x_{\varphi(\alpha)} \in L^{\varphi(\alpha)}$ ($x_{\varphi(\alpha)}$ is unique up to scalar multiples as the root spaces are one dimensional). By declaring that the lift $\tilde{\varphi} \in \mathrm{Aut}(\mathfrak{g})$ is to send x_α to $x_{\varphi(\alpha)}$, we are able to generate the action of $\tilde{\varphi}$ on all of $\mathfrak{g}$, as it may be shown that $\{x_\alpha\}_{\alpha \in \Pi}$ generates $\mathfrak{g}$. In particular, if

$$x_i \in L^{\alpha_i} \overset{\tilde{\varphi}}{\mapsto} c_{\alpha_i} x_{\varphi(\alpha_i)} \in L^{\varphi(\alpha_i)}$$

then

$$y_i \in L^{-\alpha_i} \overset{\tilde{\varphi}}{\mapsto} c_{\alpha_i}^{-1} y_{\varphi(\alpha_i)} \in L^{-\varphi(\alpha_i)}.$$

Thus, the lift is unique up to the choice of constants c_{α_i}.

2.5 Gradations of Affine Lie Algebras

Given a finite dimensional semisimple Lie algebra $\mathfrak{g}$, let $\sigma \in \mathrm{Aut}\ \mathfrak{g}$ be of order m. Thus, $\mathfrak{g}$ admits the decomposition,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{m-1}$$

where $\mathfrak{g}_j$ is the eigenspace corresponding to the eigenvalue $\exp(\frac{2\pi i j}{m})$ of σ, and

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j},$$

so that σ induces a $\mathbb{Z}_m$-gradation of $\mathfrak{g}$. The *loop algebra* over $\mathfrak{g}$ is the infinite dimensional $\mathbb{Z}$-graded Lie algebra

$$\mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}] = \bigoplus_{j \in \mathbb{Z}} \lambda^j \mathfrak{g},$$

with the Lie bracket operation defined by

$$[\lambda^i x, \lambda^j y] = \lambda^{i+j}[x, y], \text{ where } x, y \in \mathfrak{g}.$$

The automorphism σ yields a $\mathbb{Z}$-graded subalgebra, $L(\mathfrak{g}, \sigma)$, of the loop algebra, defined by

$$L(\mathfrak{g}, \sigma) = \bigoplus_{j \in \mathbb{Z}} \lambda^j \mathfrak{g}_{j \bmod m}.$$

The loop algebra is therefore $L(\mathfrak{g}, \mathrm{id})$, where id is the identity map on $\mathfrak{g}$. As shown in [28], see Theorem 2.5.3 below, every such subalgebra $L(\mathfrak{g}, \sigma)$ is isomorphic to $L(\mathfrak{g}, \nu)$, where ν is an automorphism of $\mathfrak{g}$ induced by an automorphism of the Dynkin diagram of $\mathfrak{g}$. Such automorphisms have order $k = 1, 2, 3$ only. The identity map gives rise to the *untwisted* (*i.e.* loop) algebras, while the order 2 and 3 diagram automorphisms give rise to the *twisted* algebras. Together, these algebras are generically referred to as *affine* Lie algebras.

In [28], a root space structure for $L(\mathfrak{g},\sigma)$ is developed, which is now briefly outlined. Let $\sigma \in \text{Aut}\,\mathfrak{g}$ be of order m and $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{m-1}$ the induced $\mathbb{Z}_m$-gradation of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0$, the Lie algebra of fixed points of σ. Define

$$\mathfrak{g}^{\bar{\alpha}} = \{x \in \mathfrak{g}_i : [h,x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$$

where $\bar{\alpha} = (\alpha, i)$ with $\alpha \in \mathfrak{h}^*$ and $i \in \mathbb{Z}_m$. If $\mathfrak{g}^{\bar{\alpha}} \neq 0$, then $\bar{\alpha}$ is said to be a *root* (of $\mathfrak{g}$ with respect to $\mathfrak{h}$ modulo σ). Adding roots component-wise, it follows from the Jacobi identity that

$$[\mathfrak{g}^{\bar{\alpha}}, \mathfrak{g}^{\bar{\beta}}] \subseteq \mathfrak{g}^{\bar{\alpha}+\bar{\beta}}.$$

If $\bar{\Delta}$ denotes the set of nonzero roots and $\bar{\Delta}^0$ all roots of the form $(0,i)$, $i \in \mathbb{Z}_m$, then $\mathfrak{g}$ admits the (vector space) direct sum decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\bar{\alpha}\in\bar{\Delta}} \mathfrak{g}^{\bar{\alpha}}.$$

Note that $\mathfrak{h}$ is just the zero rootspace, $\mathfrak{g}^{(0,0)}$. Furthermore, the centralizer of $\mathfrak{h}$, $\mathfrak{z}(\mathfrak{h})$, is a CSA of $\mathfrak{g}$ and decomposes as

$$\mathfrak{z}(\mathfrak{h}) = \bigoplus_{\bar{\alpha}\in\bar{\Delta}^0} \mathfrak{g}^{\bar{\alpha}}$$

so that

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{h}) \oplus \bigoplus_{\bar{\alpha}\in\bar{\Delta}\setminus\bar{\Delta}^0} \mathfrak{g}^{\bar{\alpha}}.$$

When $\bar{\alpha} \in \bar{\Delta} \setminus \bar{\Delta}^0$, the root space $\mathfrak{g}^{\bar{\alpha}}$ is one dimensional. The usual root space decomposition theory carries over. If $\sigma = \text{id}$, the standard root space structure described in §2.3 results.

The notion of roots for the algebra $L(\mathfrak{g},\sigma)$ is developed analogously. Recall that $L(\mathfrak{g},\sigma) = \bigoplus_{j\in\mathbb{Z}} L_j$, where $L_j = \lambda^j \mathfrak{g}_{j \bmod m}$. For $\alpha \in \mathfrak{h}^*$ and $j \in \mathbb{Z}$ (not $\mathbb{Z}_m$ this time), $\tilde{\alpha} = (\alpha, j)$ is called a root (of $L(\mathfrak{g},\sigma)$ with respect to $\mathfrak{h}$) if

$$L^{\tilde{\alpha}} = \{x \in L_j : [h,x] = \alpha(h)x, \forall h \in \mathfrak{h}\} \neq \{0\}.$$

Once again, adding roots component-wise, if $\tilde{\Delta}$ denotes the set of nonzero roots and $\tilde{\Delta}^0$ all roots of the form $(0,i)$, $i \in \mathbb{Z}$, then

$$[\mathfrak{g}^{\tilde{\alpha}}, \mathfrak{g}^{\tilde{\beta}}] \subseteq \mathfrak{g}^{\tilde{\alpha}+\tilde{\beta}}$$

if $\tilde{\alpha} + \tilde{\beta} \in \tilde{\Delta} \setminus \tilde{\Delta}^0$ and $L(\mathfrak{g}, \sigma)$ admits the direct sum decomposition

$$L(\mathfrak{g}, \sigma) = \mathfrak{h} \oplus \bigoplus_{\tilde{\alpha} \in \tilde{\Delta}} L^{\tilde{\alpha}}.$$

For $\tilde{\alpha} \in \tilde{\Delta} \setminus \tilde{\Delta}^0$, $L^{\tilde{\alpha}}$ is one dimensional. Also, if $\tilde{\alpha} \in \tilde{\Delta} \setminus \tilde{\Delta}^0$, then $-\tilde{\alpha} \in \tilde{\Delta} \setminus \tilde{\Delta}^0$, and

$$[L^{\tilde{\alpha}}, L^{-\tilde{\alpha}}] = \mathbb{C} h_\alpha,$$

where h_α is the unique element of $\mathfrak{h}$ with the property that

$$K(h, h_\alpha) = \alpha(h) \quad \forall h \in \mathfrak{h},$$

$K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ denoting the Killing form of $\mathfrak{g}$.[1] Similar results hold for the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ modulo σ and these may be used to prove the corresponding above results for $L(\mathfrak{g}, \sigma)$, by noting that the roots of $L(\mathfrak{g}, \sigma)$ may be mapped onto those of $\mathfrak{g}$ via the map

$$\tilde{\alpha} = (\alpha, j) \mapsto (\alpha, j \bmod m) = \bar{\alpha},$$

so that $L^{\tilde{\alpha}} = \lambda^j \mathfrak{g}^{\bar{\alpha}}$.

Choose a basis of the root system Δ_0 of $[L_0, L_0] = [\mathfrak{g}_0, \mathfrak{g}_0]$ with respect to $\mathfrak{h} \cap [L_0, L_0]$ and let Δ_0^+ be the corresponding set of positive roots (see either [28] or [33] for root systems *per se*). Define each $\alpha \in \Delta_0$ to be identically zero on the centre of L_0. Such α can thus be regarded as linear functions on $\mathfrak{h}$ and may be identified with $(\alpha, 0) \in \tilde{\Delta}^0$. Define the set of positive roots in $\tilde{\Delta}$ to be

$$\tilde{\Delta}^+ = \Delta_0^+ \cup \left\{ (\alpha, j) \in \tilde{\Delta} : j > 0 \right\}.$$

The root $\tilde{\alpha} \in \tilde{\Delta}^+$ is said to be simple if it is not the sum of any two other elements of $\tilde{\Delta}^+$.

[1] Recall that K induces a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$, namely $\langle \alpha, \beta \rangle := K(h_\alpha, h_\beta)$.

Let $\tilde{\Pi} = \{(\alpha_0, s_0), (\alpha_1, s_1), \ldots\}$ be the set of simple roots and let $\Pi = \{\alpha_0, \alpha_1, \ldots\}$. It is shown in [28] that the α_i are all distinct and that both $\tilde{\Pi}$ and Π have the same finite cardinality, say N. A few of the facts given in [28] are as follows:

Lemma 2.5.1 *(i) Each $\tilde{\alpha} \in \tilde{\Delta}$ is of the form $\tilde{\alpha} = \pm \sum_i k_i \tilde{\alpha}_i$ where $k_i \in \mathbb{Z}_+$, $\tilde{\alpha}_i \in \tilde{\Pi}$.*
(ii) $\tilde{\Pi} \subset \tilde{\Delta} \setminus \tilde{\Delta}^0$ (so that $\alpha_i \neq 0$).
(iii) Π is a linearly dependent system of vectors spanning $\mathfrak{h}^*$ and dim $\Pi = N - 1$.
(iv) If $\tilde{\alpha} \in \tilde{\Delta}^+$ is not simple, then $\tilde{\alpha} - \tilde{\alpha}_i \in \tilde{\Delta}$ for some $\tilde{\alpha}_i \in \tilde{\Pi}$.

For $0 \leq i \leq N-1$, set

$$h_i = \frac{2}{\langle \alpha_i, \alpha_i \rangle} h_{\alpha_i}$$

and choose $e_i \in L^{\tilde{\alpha}_i}, f_i \in L^{-\tilde{\alpha}_i}$ such that $[e_i, f_i] = h_i$ (note that $[L^{\tilde{\alpha}_i}, L^{-\tilde{\alpha}_i}] = \mathbb{C}h_{\alpha_i}$). Thus, $[h_i, h_j] = 0$ since CSA's are Abelian. Also, $[e_i, f_j] = \delta_{ij} h_i$ since for $i \neq j$, $[e_i, f_j] \in [L^{\tilde{\alpha}_i}, L^{-\tilde{\alpha}_j}] = 0$ since $\tilde{\alpha}_i - \tilde{\alpha}_j$ is not a root by part (i) of the lemma. Let a_{ji} be defined by $[h_i, e_j] = a_{ji} e_j$, whence $[h_i, f_j] = -a_{ji} f_j$. The matrix $A = (a_{ji})_{0 \leq i,j \leq N-1}$ is called the Cartan matrix of $L(\mathfrak{g}, \sigma)$. Note that $e_i \in L^{\tilde{\alpha}_i}$ so that

$$[h, e_i] = \alpha_i(h) e_i \in L_{s_i} \ \forall h \in \mathfrak{h}$$

where $\tilde{\alpha}_i = (\alpha_i, s_i)$ for some $\alpha_i \in \Pi$, $s_i \in \mathbb{Z}_+$. In particular, $[h_i, e_j] = \alpha_j(h_i) e_j$ and

$$\begin{aligned} \alpha_j(h_i) = K(h_i, h_{\alpha_j}) &= \frac{2}{\langle \alpha_i, \alpha_i \rangle} K(h_{\alpha_i}, h_{\alpha_j}) \\ &= \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}. \end{aligned}$$

Thus

$$a_{ji} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}.$$

It may be shown that a_{ji} is always a negative integer when $i \neq j$, which also implies that $\alpha_i \neq \alpha_j$ when $i \neq j$. Furthermore, $\{e_i, f_i, h_i : 0 \leq i \leq N-1\}$ generates $L(\mathfrak{g}, \sigma)$.

In [28] attention is now restricted to so-called indecomposable automorphisms, that is to say, those σ for which $\mathfrak{g}$ cannot be decomposed into a direct sum of σ-invariant ideals. If E is now defined to be the *real* span of Π, it follows that:

Lemma 2.5.2 *(i) The bilinear form $\langle\cdot,\cdot\rangle$ is positive definite on E and thus endows E with an inner product structure.*
(ii) dim $E = N-1$.
(iii) $\tilde{\Pi}$ is independent over $\mathbb{Z}$, i.e. if $\sum_i c_i\tilde{\alpha}_i = 0$, where $c_i \in \mathbb{Z}$, then each $c_i = 0$.

Lemma 5.10 of [28] shows that if the Cartan matrices of $L(\mathfrak{g},\sigma)$ and $L(\mathfrak{g}',\sigma')$ coincide, then there exists an isomorphism $\tilde{\psi} : L(\mathfrak{g}',\sigma') \to L(\mathfrak{g},\sigma)$ such that

$$\tilde{\psi}\left((L')^{\tau(\tilde{\alpha})}\right) = L^{\tilde{\alpha}},$$

where τ is the bijection between the simple roots guaranteed by the equality of the Cartan matrices. Moreover, if both $\mathfrak{g}$ and $\mathfrak{g}'$ are simple, then there exists an isomorphism $\psi : \mathfrak{g}' \to \mathfrak{g}$ for which $\tilde{\psi}$ is the lift, up to multiplication by a nonzero constant.

With the Cartan matrix for $L(\mathfrak{g},\sigma)$ may be associated a Dynkin diagram constructed by taking $n+1$ vertices and joining the ith and jth vertices by $a_{ij}a_{ji}$ lines. If $|a_{ij}| < |a_{ji}|$, these lines have an arrow pointing towards the ith vertex. It may be shown that every proper subdiagram of these so-called *extended* Dynkin diagrams is the disconnected union of Dynkin diagrams of the finite dimensional simple Lie algebras $\mathfrak{a}_l$, $\mathfrak{b}_l$, $\mathfrak{c}_l$, $\mathfrak{d}_l$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, $\mathfrak{g}_2$.

The root stucture of the untwisted algebras can now be developed. For $\mathfrak{g}$ simple, let σ be the identity automorphism, id. Thus $m = 1$ and $L(\mathfrak{g},\mathrm{id}) = \bigoplus_{j\in\mathbb{Z}} \lambda^i\mathfrak{g}$. Let $\alpha_1,\ldots,\alpha_n$ be the simple roots in $\Delta(\mathfrak{g},\mathfrak{h})$, the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Now let δ be the highest root. In other words, $\delta = \sum_i c_i\alpha_i$, $c_i \in \mathbb{Z}_+$, such that any other root $\sum_i c_i'\alpha_i$, $c_i' \in \mathbb{Z}_+$, has $c_i' \leq c_i$ $\forall i$. Since $L_0 = \mathfrak{g}$ and therefore $[L_0, L_0] = L_0$ (at least for the classical simple Lie algebras, which are, for all intents and purposes, the only types under consideration here), it follows that Δ_0^+ is the $\mathbb{Z}_+$-span of $\{(\alpha_1,0),\ldots,(\alpha_n,0)\}$, since each $(\alpha_i,0)$ is a simple root of $L(\mathfrak{g},\mathrm{id})$ with respect to $\mathfrak{h}$. By definition of Δ_+, $(-\delta,1)$ is also a positive root and is moreover simple. If this were not the case, then there would exist simple roots $(\alpha,0),(\beta,1)$, with α positive, such that $(-\delta,1) = (\alpha,0)+(\beta,1)$, so that $\delta+\alpha = -\beta$ is still a root, contradicting the maximality of δ. The simple roots of $L(\mathfrak{g}, \mathrm{id})$ are thus

$$\tilde{\Pi} = \{(-\delta, 1), (\alpha_1, 0), \ldots, (\alpha_n, 0)\}.$$

Note that $\tilde{\Pi}$ has the correct cardinality since $\Pi = \{-\delta, \alpha_1, \ldots, \alpha_n\}$ has dimension $n = N-1$, where N is the cardinality of Π, since the original simple roots in $\Delta(\mathfrak{g}, \mathfrak{h})$ are, by definition, linearly independent. The untwisted algebras over $\mathfrak{g}$ are denoted by $\mathfrak{g}^{(1)}$.

As noted above, it is enough to consider the algebras $L(\mathfrak{g}, \nu)$, where ν is an automorphism of $\mathfrak{g}$ induced by an automorphism of its Dynkin diagram. The twisted algebras are given by $L(\mathfrak{g}, \nu)$, where ν has order 2 (possible for the algebras $\mathfrak{a}_l, \mathfrak{d}_l, \mathfrak{e}_6$) or order 3 (only possible for $\mathfrak{d}_4$). The twisted algbras are denoted by $\mathfrak{g}^{(k)}$, k being the order of the automorphism ν.

Before proceeding any further, the root structure of the twisted algebra over $\mathfrak{sl}(3) \cong \mathfrak{a}_2$ is discussed. The algebra in question is $L(\mathfrak{a}_2, \nu)$ where ν is generated by the Dynkin diagram automorphism of order 2. The algebra $\mathfrak{a}_2$ has canonical generators x_i, y_i, h_i, $i = 1, 2$, and simple roots α_1, α_2. The Dynkin diagram is just $\circ - \circ$, and the order 2 automorphism, ν, just swaps the vertices, each corresponding to α_i. Thus, ν just interchanges the labels 1 and 2. In other words, the induced automorphism (also denoted by ν) is generated by

$$\nu(x_i) = x_{\nu(i)},\ \nu(y_i) = y_{\nu(i)},\ \nu(h_i) = h_{\nu(i)},\ i = 1, 2,$$

where $\nu(i) = 3 - i$. Hence, under ν, the x_i are swapped, likewise the y_i, and also the h_i. Since the simple roots α_i are interchanged, the basis element, y_0, of the one dimensional root space for the highest root, $\delta := \alpha_1 + \alpha_2$, gets mapped to $-y_0$ and similarly for $x_0 \in L^{-\delta}$. This follows from the fact that $[x_1, x_2] \subset \mathfrak{g}^{\alpha_1+\alpha_2}$. In fact,

$$y_0 = [x_1, x_2] \overset{\nu}{\mapsto} [x_2, x_1] = -y_0.$$

In the standard representation consisting of 3×3 trace free matrices for $\mathfrak{sl}(3)$,

$h_1 = \mathrm{diag}(1,-1,0)$, $h_2 = \mathrm{diag}(0,1,-1)$, while

$$x_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and the remaining elements are given by

$$y_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, x_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Under ν, $\mathfrak{a}_2 = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ consists of the fixed points of ν, so that $\mathfrak{g}_0$ is generated by $\bar{x}, \bar{y}, \bar{h}$, where

$$\bar{x} = x_1 + x_2,\ \bar{y} = 2(y_1 + y_2),\ \bar{h} = 2(h_1 + h_2).$$

Under this identification, $\mathfrak{g}_0 \cong \mathfrak{b}_1 \cong \mathfrak{sl}(2)$. More generally, for $\mathfrak{g} = \mathfrak{a}_{2n}$, the diagram automorphism $\nu(i) = 2n - i + 1$ gives $\mathfrak{g}_0 \cong \mathfrak{b}_n$. Thus,

$$\mathfrak{g}_0 = \mathrm{Sp}\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

Also, $\mathfrak{g}_1$, the eigenspace corresponding to the eigenvalue -1 of ν, is spanned by $h_1 - h_2, x_1 - x_2, y_1 - y_2, y_0, x_0$. That is,

$$\mathfrak{g}_1 = \mathrm{Sp}\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}.$$

The twisted algebra over $\mathfrak{sl}(3)$ is therefore represented by

$$L(\mathfrak{a}_2, \nu) = \bigoplus_{j \in \mathbb{Z}} \lambda^j \mathfrak{a}_{2_{j \bmod 2}} = \left\langle \lambda^{2j} \otimes \mathfrak{g}_0 : j \in \mathbb{Z} \right\rangle \oplus \left\langle \lambda^{2j+1} \otimes \mathfrak{g}_1 : j \in \mathbb{Z} \right\rangle.$$

The algebra $\mathfrak{g}_0$ has CSA $\mathfrak{h} = \mathrm{Sp}\{\bar{h}\}$ and $\mathfrak{z}(\mathfrak{h}) = \mathrm{Sp}\{h_1, h_2\}$ is a CSA for $\mathfrak{a}_2$ itself. In [28], it is shown that the roots of $\mathfrak{a}_2$ satisfy $\alpha_i\mid_{\mathfrak{h}} = \bar{\alpha}$, $i = 1, 2$, where $\bar{\alpha}$ is the simple root for $\mathfrak{g}_0 \cong \mathfrak{sl}(2)$. ($\mathrm{Sp}\{\bar{x}\}$ is the root space for $\bar{\alpha}$.) Furthermore, $(\bar{\alpha}, 0)$ is a simple root for $L(\mathfrak{a}_2, \nu)$ with respect to $\mathfrak{h} = \mathrm{Sp}\{\bar{h}\}$. Let δ be as previously defined, that is, $\delta = \alpha_1 + \alpha_2$, the highest root in $\Delta(\mathfrak{a}_2, \mathfrak{z}(\mathfrak{h}))$. Introduce δ^* as follows:

$$\delta^* := \delta\mid_{\mathfrak{h}} = \alpha_1\mid_{\mathfrak{h}} + \alpha_2\mid_{\mathfrak{h}} = 2\bar{\alpha}.$$

Then $(-\delta^*, 1)$ may be shown to be a simple root for $\Delta(L(\mathfrak{a}_2, \nu), \mathfrak{h})$. Thus the simple roots of $L(\mathfrak{a}_2, \nu)$ with respect to $\mathfrak{h}$ are

$$\tilde{\alpha}_0 = (-2\bar{\alpha}, 1) \text{ and } \tilde{\alpha}_1 = (\bar{\alpha}, 0).$$

The corresponding root spaces are

$$L^{\tilde{\alpha}_1} = \mathrm{Sp}\{\bar{x}\} = \mathrm{Sp}\left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right\}$$

and, as shown in [28],

$$L^{\tilde{\alpha}_0} = \lambda(\mathfrak{a}_2^{-\delta} \cap \mathfrak{g}_1) = \mathrm{Sp}\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}\right\}.$$

Note that in the untwisted case, $\mathfrak{g}_0$ is the whole base algebra, since ν is then the identity automorphism, whereas in the twisted case, $\mathfrak{g}_0$ is a subalgebra of the base algebra. This accounts for the smaller number of simple roots in the twisted case. For example, for the twisted algebra over $\mathfrak{sl}(3)$, the simple roots are $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ as above. On the other hand, for the untwisted case, the roots are just $(-\delta, 1), (\alpha_1, 0), (\alpha_2, 0)$, as already discussed, where the α_i are just the roots for $\mathfrak{a}_2$ and $\delta = \alpha_1 + \alpha_2$. This is what is behind the fact that typically, when associating pde's to these two algebras, the twisted algebra yields a single equation while the untwisted algebra yields a system of two equations.

It is now possible to define certain types of gradations on the algebras $L(\mathfrak{g},\nu)$, where ν is the automorphism induced from that of the Dynkin diagram of $\mathfrak{g}$, so that ν necessarily has order $k = 1, 2, 3$. The simple roots of $L(\mathfrak{g},\nu)$ are $\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$, where $\tilde{\alpha}_0 = (\alpha_0, 1)$ is the lowest root of the form $(\alpha, 1)$, and $\tilde{\alpha}_i = (\alpha_i, 0)$ for $1 \leq i \leq n$. As usual, α_i, $1 \leq i \leq n$, are the simple positive roots of $\mathfrak{g}_0$ with respect to its CSA, $\mathfrak{h}$.

Consider any sequence $s_0, s_1, \ldots, s_n$ of nonnegative integers, not all zero. The $\mathbb{Z}$-independence of $\tilde{\Pi} = \{\tilde{\alpha}_0, \ldots, \tilde{\alpha}_n\}$ implies that any root $\tilde{\alpha}$ may be uniquely written in the form

$$\tilde{\alpha} = \sum_{i=0}^{n} k_i \tilde{\alpha}_i, \;\; k_i \in \mathbb{Z}.$$

Thus, each root $\tilde{\alpha}$ may be represented by a row vector $\mathbf{k} = (k_0, \ldots, k_n)$. Assign a degree to each $\tilde{\alpha}$ as follows. Let

$$\deg_{\mathbf{s}} \tilde{\alpha} = \sum_{i=0}^{n} k_i s_i$$

be the degree with respect to $\mathbf{s}$. That is, $\deg_{\mathbf{s}} \tilde{\alpha} = \mathbf{k}.\mathbf{s}$, where $\mathbf{s} = (s_0, \ldots, s_n)$. Thus, $\deg_{\mathbf{s}} \tilde{\alpha}_i = s_i$. Note that the vector $\mathbf{k}$ consists of entries which are either all nonnegative or all nonpositive, by virtue of the $\tilde{\alpha}_i$ being simple roots. Furthermore, since all the s_i are, by definition, nonnegative, it follows that a sufficient condition that $\deg \tilde{\alpha} = 0$ is that $\tilde{\alpha}$ be the zero root for which the corresponding root space is the CSA, $\mathfrak{h}$, of $\mathfrak{g}_0$. The so-called *gradation of type* $\mathbf{s} = (s_0, \ldots, s_n)$, is defined to be

$$L(\mathfrak{g},\nu) = \bigoplus_{j\in\mathbb{Z}} L_j$$

where

$$L_j := \sum_{\deg_{\mathbf{s}} \tilde{\alpha} = j} L^{\tilde{\alpha}}.$$

Thus, the elements of the root space corresponding to a particular root are assigned the degree of that root. From the above remarks, $\mathfrak{h} \subseteq L_0$, irrespective of $\mathbf{s}$. The fact that the above construction does produce a genuine gradation of $L(\mathfrak{g},\nu)$ follows from the standard properties of root spaces. If $X \in L^{\tilde{\alpha}}$ and $Y \in L^{\tilde{\beta}}$ with $\deg \tilde{\alpha} = i$

and deg $\tilde{\beta} = j$, then $[X, Y] \in L^{\tilde{\alpha}+\tilde{\beta}}$, provided $\tilde{\alpha}+\tilde{\beta}$ is a root (otherwise $[X, Y] = 0$). From the definition of degree, deg $(\tilde{\alpha} + \tilde{\beta})$ = deg $\tilde{\alpha}$ + deg $\tilde{\beta}$ = $i + j$. Therefore, $[L_i, L_j] \subseteq L_{i+j}$. Hence, s induces a *bona fide* gradation of $L(\mathfrak{g}, \nu)$.

The original decomposition,

$$L(\mathfrak{g}, \nu) = \bigoplus_{i \in \mathbb{Z}} \lambda^i \mathfrak{g}_{i \bmod k},$$

corresponds to the gradation of type $\mathbf{s} = (1, 0, \ldots, 0)$ and is called the *homogeneous gradation.* In this case, if

$$\tilde{\alpha} = \sum_{i=0}^{n} k_i \tilde{\alpha}_i = \left(\sum_{i=0}^{n} k_i \alpha_i, k_0 \right),$$

then $\deg_{\mathbf{s}} \tilde{\alpha} = \mathbf{k}.\mathbf{s} = k_0$. Thus, if $\deg_{\mathbf{s}} \tilde{\alpha} = j$, then

$$\tilde{\alpha} = j(\alpha_0, 1) + \left(\sum_{i=1}^{n} k_i \alpha_i, 0 \right).$$

By definition, α_0 is the lowest root, so that $\alpha_0 = \sum_{m=1}^{n} (-l_m) \alpha_m$, where $l_m \geq 0$ (α_m, $m = 1 \ldots n$, being simple roots). Thus, for $i = 1 \ldots n$,

$$\begin{aligned} (\alpha_i, j) &= j(\alpha_0, 1) + \sum_{m=1}^{n} j l_m (\alpha_m, 0) + (\alpha_i, 0) \\ &= j\tilde{\alpha}_0 + \sum_{\substack{m=1 \\ m \neq i}}^{n} j l_m \tilde{\alpha}_m + (j l_i + 1)\tilde{\alpha}_i. \end{aligned}$$

This gives the unique decomposition of the root $(\alpha_i, 0)$ in the form $\sum_{i=0}^{n} k_i \tilde{\alpha}_i$, with $k_0 = j$. In particular, $\deg_{\mathbf{s}}(\alpha_i, j) = j$ for all $i = 1 \ldots n$. Similarly, $(0, j) = j\tilde{\alpha}_0 + \sum_{m=1}^{n} j l_m \tilde{\alpha}_m$, so that $\deg_{\mathbf{s}}(0, j) = j$. It follows that

$$\begin{aligned} L^j \supseteq L^{(0,j)} \bigoplus_{i=1}^{n} L^{(\alpha_i, j)} &= \lambda^j \otimes \left(\left(\mathfrak{z}(\mathfrak{h}) \bigoplus_{\tilde{\alpha} \in \tilde{\Delta} \setminus \tilde{\Delta}^0} L^{\tilde{\alpha}} \right) \cap \mathfrak{g}_{j \bmod k} \right) \\ &= \lambda^j \otimes (\mathfrak{g} \cap \mathfrak{g}_{j \bmod k}) \\ &= \lambda^j \otimes \mathfrak{g}_{j \bmod k}. \end{aligned}$$

Hence, for the homogeneous gradation, $L^j = \lambda^j \mathfrak{g}_{j \bmod k}$ as claimed.

Without loss of generality, it suffices to look at only those $\mathbf{s} = (s_0, \ldots, s_n)$ for which the s_i have no nontrivial common factor, as the gradations produced by $\mathbf{s}_1$ and $\mathbf{s}_2 = p\mathbf{s}_1$, where p is a positive integer, are isomorphic.

As mentioned before, it is enough to consider the algebras $L(\mathfrak{g}, \nu)$, where ν is the automorphism induced from a Dynkin diagram automorphism. In fact, the following is true [28]:

Theorem 2.5.3 *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and σ an automorphism of finite order. Then there exists an automorphism, ν, of $\mathfrak{g}$, induced by an automorphism of its Dynkin diagram and a $\mathbb{Z}$-gradation of $L(\mathfrak{g}, \nu)$ of type $\mathbf{s} = (s_0, \ldots, s_n)$, such that $L(\mathfrak{g}, \nu)$ is isomorphic to the $\mathbb{Z}$-graded Lie algebra $L(\mathfrak{g}, \sigma)$ by an isomorphism under which the two $\mathbb{Z}$-gradations correspond.*

Theorem 5.15 of Chapter X of [28] shows how to generate all automorphisms of a given finite order of $\mathfrak{g}$ from diagram automorphisms and gradations of type $\mathbf{s}$. This may be used to classify automorphisms of given order up to conjugacy. To illustrate these ideas, some examples of gradations of algebras over $\mathfrak{a}_2 \cong \mathfrak{sl}(3)$ are given.

First, consider the untwisted case, $L(\mathfrak{a}_2, \mathrm{id})$. The roots for $\mathfrak{sl}(3)$ and their corresponding root spaces are as follows:

$$\alpha_1 \leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \delta \leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$-\alpha_1 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad -\alpha_2 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad -\delta \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

To $\alpha = 0$ corresponds the CSA, $\mathfrak{h}$, of diagonal matrices. The simple roots for $\mathfrak{sl}(3)$ are α_1, α_2, and the highest root is $\delta = \alpha_1 + \alpha_2$, so that the simple roots for $L(\mathfrak{a}_2, \mathrm{id})$ are $\tilde{\alpha}_0 = (-\delta, 1), \tilde{\alpha}_1 = (\alpha_1, 0), \tilde{\alpha}_2 = (\alpha_2, 0)$. Now, a typical root in $\Delta(L(\mathfrak{a}_2, \mathrm{id}), \mathfrak{h})$ is

$$\tilde{\alpha} = \sum_{i=0}^{2} k_i \tilde{\alpha}_i = ((k_1 - n)\alpha_1 + (k_2 - n)\alpha_2, n) = (\alpha, n),$$

where $n = k_0$. The corresponding root space is $\lambda^n \otimes L^\alpha$, where L^α is the root space for α in $\mathfrak{a}_2$. The degrees for the three gradations $\mathsf{s}_1 = (1,0,0)$ (the homogeneous gradation), $\mathsf{s}_2 = (1,1,0)$ (an "intermediate" gradation) and $\mathsf{s}_3 = (1,1,1)$ (known in the literature as the *principal* gradation), are listed in Table 2.1. The entries in the second column correspond to the $\mathfrak{a}_2$-root, $\alpha = (k_1 - n)\alpha_1 + (k_2 - n)\alpha_2$, arising from $\tilde{\alpha} = (\alpha, n)$ as above.

As can be seen, the s_2-gradation essentially breaks up the algebra into two parts consisting of elements of even degree, namely

$$\lambda^n \otimes \;\mathfrak{h} \oplus \mathrm{Sp}\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right\},$$

and those of odd degree, namely

$$\lambda^n \otimes \mathrm{Sp}\left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}\right\}.$$

On the other hand, the principal gradation gives a mod 3 splitting of the algebra. The elements of degree 0 mod 3 are $\lambda^n \otimes \mathfrak{h}$, those of degree 1 mod 3 are

$$\lambda^n \otimes \mathrm{Sp}\left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}\right\},$$

and those of degree 2 mod 3 are

$$\lambda^n \otimes \mathrm{Sp}\left\{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix}\right\}.$$

Next, consider the twisted case, $L(\mathfrak{a}_2, \nu)$, where ν is generated by the Dynkin diagram automorphism of order 2. Bases for $\mathfrak{g}_0$ and $\mathfrak{g}_1$ have previously been given.

Table 2.1: Some gradations of $\mathfrak{a}_2^{(1)}$

Root space	α	$\mathbf{k}$	Degree		
			$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$
$\lambda^n \mathfrak{h}$	0	(n,n,n)	n	$2n$	$3n$
$\lambda^n \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	α_1	$(n,n+1,n)$	n	$2n+1$	$3n+1$
$\lambda^n \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	α_2	$(n,n,n+1)$	n	$2n$	$3n+1$
$\lambda^n \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	δ	$(n,n+1,n+1)$	n	$2n+1$	$3n+2$
$\lambda^n \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$-\alpha_1$	$(n,n-1,n)$	n	$2n-1$	$3n-1$
$\lambda^n \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$-\alpha_2$	$(n,n,n-1)$	n	$2n$	$3n-1$
$\lambda^n \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$-\delta$	$(n,n-1,n-1)$	n	$2n-1$	$3n-2$

Recall that the simple roots are $\tilde{\alpha}_0 = (-2\bar{\alpha}, 1)$ and $\tilde{\alpha}_1 = (\bar{\alpha}, 0)$, where $\bar{\alpha}$ is the simple root for $\mathfrak{g}_0 \cong \mathfrak{sl}(2)$. The basic root spaces are

$$
\begin{aligned}
L^{\tilde{\alpha}_1} &= \mathrm{Sp}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & L^{-\tilde{\alpha}_1} &= \mathrm{Sp}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
L^{\tilde{\alpha}_0} &= \mathrm{Sp}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}, & L^{-\tilde{\alpha}_0} &= \mathrm{Sp}\begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
L^{\tilde{\alpha}_0+\tilde{\alpha}_1} &= \mathrm{Sp}\begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & -\lambda & 0 \end{pmatrix}, & L^{-\tilde{\alpha}_0-\tilde{\alpha}_1} &= \mathrm{Sp}\begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & -\lambda \\ 0 & 0 & 0 \end{pmatrix}, \\
L^{(0,0)} &= \mathrm{Sp}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & L^{(0,1)} &= \mathrm{Sp}\begin{pmatrix} \lambda & 0 & 0 \\ 0 & -2\lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}.
\end{aligned}
$$

The homogeneous and principal gradations, $\mathbf{s}_1 = (1,0)$ and $\mathbf{s}_2 = (1,1)$ respectively, are summarised in Table 2.2, in which $\tilde{\alpha} = k_0\tilde{\alpha}_0 + k_1\tilde{\alpha}_1 = ((k_1 - 2k_0)\bar{\alpha}, k_0)$.

Table 2.2: Some gradations of $\mathfrak{a}_2^{(2)}$

Root space	α	k	Degree s_1	Degree s_2
$\lambda^{2n}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$	$(0,2n)$	$(2n,4n)$	$2n$	$6n$
$\lambda^{2n+1}\begin{pmatrix}1&0&0\\0&-2&0\\0&0&1\end{pmatrix}$	$(0,2n+1)$	$(2n+1,4n+2)$	$2n+1$	$6n+3$
$\lambda^{2n}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$	$(\bar{\alpha},2n)$	$(2n,4n+1)$	$2n$	$6n+1$
$\lambda^{2n}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$	$(-\bar{\alpha},2n)$	$(2n,4n-1)$	$2n$	$6n-1$
$\lambda^{2n+1}\begin{pmatrix}0&1&0\\0&0&-1\\0&0&0\end{pmatrix}$	$(\bar{\alpha},2n+1)$	$(2n+1,4n+3)$	$2n+1$	$6n+4$
$\lambda^{2n+1}\begin{pmatrix}0&0&0\\1&0&0\\0&-1&0\end{pmatrix}$	$(-\bar{\alpha},2n+1)$	$(2n+1,4n+1)$	$2n+1$	$6n+2$
$\lambda^{2n+1}\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$	$(2\bar{\alpha},2n+1)$	$(2n+1,4n+4)$	$2n+1$	$6n+5$
$\lambda^{2n+1}\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$	$(-2\bar{\alpha},2n+1)$	$(2n+1,4n)$	$2n+1$	$6n+1$

Chapter 3

Weyl Groups and Heisenberg Subalgebras

This chapter presents the link between the conjugacy classes of the Weyl group and the collection of all possible Heisenberg subalgebras of any given affine Lie algebra. In §3.1, the Weyl group is defined and the conjugacy classes for $\mathfrak{a}_n$ described, with specific examples for $\mathfrak{a}_2$ and $\mathfrak{a}_3$. This follows Delduc and Fehér [14, 15]. In §3.2, the lift of the Weyl group to a subgroup of inner automorphisms of a Lie algebra is discussed. Specific examples are once again given for $\mathfrak{a}_2$ and $\mathfrak{a}_3$. The Kač-Helgason theorem [35, 28] on the classification of automorphisms of finite order of simple, finite dimensional Lie algebras is presented in §3.3. Subsequently, the notion of a shift vector is introduced [3] and related to the fundamental dominant weights [33] in order to determine the $\mathsf{s}[w]$-vectors associated to a given conjugacy class of the Weyl group with representative w. The Bos algorithm that relates these vectors to those of the Kač-Helgason theorem (Theorem 3.3.1) is mentioned. In §3.4, Heisenberg subalgebras are defined and the link established with conjugacy classes of the Weyl group. Examples are given for $\mathfrak{a}_2$ and $\mathfrak{a}_3$. Finally, in §3.5, a few remarks are made about the specific representation of Heisenberg subalgebras desired for later use.

3.1 The Weyl Group and its Conjugacy Classes

Consider a finite dimensional simple Lie algebra, $\mathfrak{g}$, over $\mathbb{C}$, with Cartan subalgebra, $\mathfrak{h}$. Let Δ denote the set of roots of $\mathfrak{g}$. Recall that the bilinear form, $\langle\cdot,\cdot\rangle$, on Δ, induced by the Killing form, K, is actually an inner product on $\mathfrak{h}^*$. Thus, to each nonzero root α, one may associate a reflection $r_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ which, by definition, is a linear transformation leaving a hyperplane pointwise fixed and mapping any element of $\mathfrak{h}^*$ orthogonal to that hyperplane to its negative. The hyperplane consists of all those $\gamma \in \mathfrak{h}^*$ for which $\langle\gamma,\alpha\rangle = 0$. It follows that

$$r_\alpha(\beta) = \beta - \frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\,\alpha, \ \forall\beta \in \Delta.$$

Note that r_α^2 is the identity.

The Weyl group of $\mathfrak{g}$ is denoted $W(\mathfrak{g})$ and is defined to be the group generated by all the reflections r_α, where $\alpha \in \Delta$. In fact, the so-called simple reflections r_α, where α is a simple root, will suffice to generate $W(\mathfrak{g})$ which is a normal subgroup of $\mathrm{Aut}(\Delta)$. In fact, $\mathrm{Aut}(\Delta)$ is the semidirect product of $W(\mathfrak{g})$ and the group of Dynkin diagram symmetries.

Under the Killing form identification of $\mathfrak{h}$ and $\mathfrak{h}^*$, $W(\mathfrak{g})$ may be thought of as a map $\mathfrak{h} \to \mathfrak{h}$. Given a root α,

$$r_\alpha(h) = h - \frac{2K(h,h_\alpha)}{K(h_\alpha,h_\alpha)}\,h_\alpha, \quad \forall h \in \mathfrak{h},$$

where $h_\alpha \in \mathfrak{h}$ corresponds to $\alpha \in \mathfrak{h}^*$. In the case of the simple roots, α_i, with corresponding $h_i \in \mathfrak{h}$,

$$r_{\alpha_i}(h_j) = h_j - a_{ij}h_i,$$

where (a_{ij}) is the Cartan matrix for $\mathfrak{g}$, that is $a_{ij} = 2\langle\alpha_j,\alpha_i\rangle/\langle\alpha_i,\alpha_i\rangle$. Allowing for the abuse of notation, $r_\alpha : \mathfrak{h} \to \mathfrak{h}$ is just the adjoint (transpose) of $r_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$.

We shall only need to distinguish between Weyl group elements that are not conjugate. Recall that a *conjugacy class* of a group G, containing the element $a \in G$, is defined to be

$$\{b \in G : \exists c \in G, \ b = cac^{-1}\}.$$

There is a special conjugacy class of $W(\mathfrak{g})$, known as the *Coxeter class*, of which a typical element is the product of reflections $\prod_{\alpha\in\Delta} r_\alpha$, where α ranges over *all* elements of Δ. All such elements are conjugate, irrespective of the base, Δ, chosen or the order in which the reflections occur [11]. In particular, for the standard choice of simple roots $\{\alpha_j : 1 \leq j \leq n\}$, we define the Coxter element to be

$$w_C := r_{\alpha_1} r_{\alpha_2} \dots r_{\alpha_n},$$

and shall use it as our representative of the Coxeter class.

The Weyl groups of the simple Lie algebras have long been known. In particular, for $\mathfrak{a}_n$, it may be shown [33] that there is a homomorphism taking r_{α_i}, $i = 1, \dots, n$, to the transposition $(i, i+1)$ (entries mod n) of S_{n+1}, the permutation group of $\{0, 1, \dots, n\}$. Since the r_{α_i} generate $W(\mathfrak{a}_n)$ and the transpositions $(i, i+1)$ generate S_{n+1}, it follows that

$$W(\mathfrak{a}_n) \cong S_{n+1}.$$

The conjugacy classes of $W(\mathfrak{g})$ are in one-to-one correspondence with what are known as the Heisenberg subalgebras of $\mathfrak{g}$, up to isomorphism [37]. The precise definition of a Heisenberg subalgebra is postponed until §3.4. The conjugacy classes have been classified for the simple algebras by Carter [9]. For the case $\mathfrak{g} = \mathfrak{a}_n$, this amounts to looking at the conjugacy classes of S_{n+1}. As is well known, there is a bijection between these and the partitions of the integer $n+1$. By *partition* is meant a sequence of positive integers $\{m_1, m_2, \dots, m_r\}$, with $n+1 \geq m_1 \geq m_2 \geq \dots \geq m_r \geq 1$, such that

$$m_1 + m_2 + \cdots + m_r = n + 1.$$

Carter [9] has shown that every $w \in W(\mathfrak{g})$ may be decomposed into a product of reflections in reduced form (*i.e.* entailing as few r_α, not necessarily simple, as possible, which is the case if and only if all the α are linearly independent):

$$w = \left(\prod_{\beta\in J_1} r_\beta\right)\left(\prod_{\gamma\in J_2} r_\gamma\right)$$

where $J_1 \cap J_2 = \emptyset$ and the roots in J_k, $k = 1, 2$, are mutually orthogonal, which means the ordering in J_1 and J_2 is irrelevant. To this decomposition one can associate a so-called *Carter diagram* as follows: to each r_α, $\alpha \in J_1 \cup J_2$, assign a node, and for each pair of roots r_β, r_γ, $\beta \in J_1, \gamma \in J_2$, join the corresponding nodes by k lines, where

$$k = \frac{4\langle \beta, \gamma \rangle^2}{\langle \beta, \beta \rangle \langle \gamma, \gamma \rangle} = 4\cos^2\theta,$$

and θ is the angle between β and γ. (Amongst each of J_1, J_2, there are no lines joining the nodes as the roots are mutually orthogonal.) The Carter diagram for the conjugacy class consisting of the identity element is just the empty set, $\emptyset$.

As shown in [9], conjugate elements in $W(\mathfrak{g})$ give rise to the same Carter diagram, though the diagram does not uniquely determine the conjugacy class. For $\mathfrak{a}_n$, however, the Carter diagrams *do* precisely determine the conjugacy classes. In fact, provided there are no cycles in the Carter diagram, which is indeed the case for $\mathfrak{a}_n$, the Carter diagram is just the Dynkin diagram of some Weyl subgroup of $W(\mathfrak{g})$. This is explained in some detail in [9]. Furthermore, these diagrams may be shown to correspond to the conjugacy classes of *primitive* elements in the regular subalgebras of $\mathfrak{g}$, which were classified in [19] and may be found from the extended Dynkin diagram of $\mathfrak{g}$. An element $w \in W(\mathfrak{g})$ is said to be primitive if $\det(\mathrm{id} - w) = \det A$, where $A = (a_{ij})$ is the Cartan matrix of $\mathfrak{g}$. Delduc and Fehér [14] exploit this fact to construct the Heisenberg subalgebras for the classical Lie algebras and $\mathfrak{g}_2$.

It should be emphasised that for $\mathfrak{a}_n$, the Carter diagram is merely a disconnected sequence of Dynkin diagrams of subalgebras of $\mathfrak{a}_n$. However, for general simple algebras, the Carter diagrams possess connected components containing cycles, unlike any of the Dynkin diagrams of simple algebras.

In view of the observations of the preceding paragraph, it is necessary to examine the primitive elements of $W(\mathfrak{g})$ and its Weyl subgroups. This has been extensively done by Delduc and Fehér [14, 15] for those conjugacy classes whose representatives admit a regular eigenvector, that is, an element of $\mathfrak{h}$ whose adjoint map annihilates $\mathfrak{h}$ and nothing more.

The conjugacy classes of $\mathfrak{a}_n$ that admit a regular eigenvector are neatly described in terms of their analogous partitions of $n+1$. It is shown in §3.1 of [14] that only those partitions, $\mathcal{P}$, of the form $\mathcal{P} = \{p, \cdots, p\}$, containing s elements, say, so that $ps = n+1$, or $\mathcal{P} = \{p, \cdots, p, 1\}$, containing s elements, so that $p(s-1)+1 = n+1$, correspond to conjugacy classes of $W(\mathfrak{a}_n)$ possessing regular eigenvectors, which are subsequently listed. Consequently, for larger values of n, a given conjugacy class will more often than not admit no regular eigenvectors at all.

Furthermore, as Delduc and Fehér note in §4.1 of [14], the work of [19] and [9] jointly shows that, at least in the cases of the classical Lie algebras and $\mathfrak{g}_2$, for each conjugacy class of $W(\mathfrak{g})$, there may be chosen a representative of the form

$$w = w_1 w_2 \dots w_r,$$

where each w_k is a primitive element of $W(\mathfrak{g}_k)$, and $\mathfrak{g}_k$ is a simple factor of the regular semisimple subalgebra

$$\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r \subset \mathfrak{g}.$$

The diagram obtained by the (unjoined) Dynkin diagrams of $\mathfrak{g}_1, \mathfrak{g}_2, \dots, \mathfrak{g}_r$, is the Carter diagram of the conjugacy class of w. Moreover, the Cartan subalgebra, $\mathfrak{h}$, has the corresponding decomposition,

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_r \oplus \mathfrak{h}',$$

such that $\mathfrak{h}_k$ is a CSA of $\mathfrak{g}_k$, and $\mathfrak{h}'$ is the subspace consisting of the fixed points of w. The dimension of $\mathfrak{h}'$ is given by rank $\mathfrak{g} - \sum_{k=1}^{r}$ rank $\mathfrak{g}_k$, which is the same as the difference between the number of nodes in the Dynkin diagram of $\mathfrak{g}$ and the Carter diagram for the conjugacy class of $w = w_1 w_2 \dots w_r$.

For the simple Lie algebras, the Coxeter elements are always primitive, so that there is always at least one primitive conjugacy class. The Carter diagram representing the Coxeter class is merely the Dynkin diagram of $\mathfrak{g}$.

In the case of the Lie algebras of type $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and the exceptional algebra, $\mathfrak{g}_2$, the Coxeter class is, in fact, the only primitive class. For the algebras of type $\mathfrak{d}$, the situation is a little more complicated. Following [14, 15], we shall not concern ourselves with the remaining exceptional algebras, although those of type $\mathfrak{e}$ are discussed in [3].

From [19], we know that the regular subalgebras of $\mathfrak{a}_n$ are

$$\mathfrak{a}_{n_1} \oplus \mathfrak{a}_{n_2} \cdots \oplus \mathfrak{a}_{n_r},$$

where $\sum_{i=1}^{r}(n_i+1) = n+1$, and $n_1 \geq n_2 \geq \ldots \geq n_r \geq 0$, with $\mathfrak{a}_0 := \{0\}$. The associated class representative is then $w = w_1 w_2 \ldots w_r$, where w_i is the Coxeter element of $\mathfrak{a}_{n_i}$, since these constitute the only candidates up to conjugacy for the primitive elements of $\mathfrak{a}_{n_i}$. This is summarised in Table 4 of [14, §4.1]. The results for those conjugacy classes of $\mathfrak{a}_n$ admitting a regular eigenvector are reproduced here: For the partition $\mathcal{P} = \{p, \ldots, p\}$ of $n+1$ with $n = ps-1$, the regular subalgebra is

$$\mathfrak{a}_{p-1} \oplus \cdots \oplus \mathfrak{a}_{p-1}$$

(s copies) and the CSA, $\mathfrak{h}$, decomposes as

$$\mathfrak{h} = \mathfrak{h}_{p-1} \oplus \cdots \oplus \mathfrak{h}_{p-1} \oplus \mathfrak{h}'_{s-1},$$

where $\mathfrak{h}_{p-1}$ denotes the CSA of $\mathfrak{a}_{p-1}$, and $\mathfrak{h}'_{s-1}$ is the subspace of fixed points of w, with

$$\begin{aligned} \dim \mathfrak{h}'_{s-1} &= \text{rank } \mathfrak{a}_{ps-1} - s \text{ rank } \mathfrak{a}_{p-1} \\ &= ps - 1 - s(p-1) = s - 1. \end{aligned}$$

For the remaining case, that is, the partition $\mathcal{P} = \{p, \ldots, p, 1\}$ of $n+1$ with $n = p(s-1)$, the regular subalgebra is

$$\mathfrak{a}_{p-1} \oplus \cdots \oplus \mathfrak{a}_{p-1}$$

($s-1$ copies) with

$$\mathfrak{h} = \mathfrak{h}_{p-1} \oplus \cdots \oplus \mathfrak{h}_{p-1} \oplus \mathfrak{h}'_{s-1},$$

where, once again,

$$\begin{aligned}\dim \mathfrak{h}'_{s-1} &= \text{rank } \mathfrak{a}_{p(s-1)} - (s-1) \text{ rank } \mathfrak{a}_{p-1} \\ &= p(s-1) - (s-1)(p-1) = s-1.\end{aligned}$$

Example 3.1.1 *The conjugacy classes of* $\mathfrak{a}_2$ *and* $\mathfrak{a}_3$.

Take the standard representation of $\mathfrak{a}_n$ consisting of $(n+1) \times (n+1)$ trace free matrices over $\mathbb{C}$. The CSA, $\mathfrak{h}$, consists of the diagonal matrices, of which we define the elements $h_i := E_{i,i} - E_{i+1,i+1}$, $i = 0, 1, \ldots, n$, where $E_{i,j}$ is the elementary matrix with 1 in the (i,j)-entry and 0 elsewhere. Note that $\{h_i : i = 1, \ldots, n\}$ forms a basis for $\mathfrak{h}$ and $h_0 = -(h_1 + \cdots + h_n)$. The h_i, $i = 1, \ldots, n$, (strictly speaking, certain scalar multiples of the h_i) correspond under the Killing form identification to the simple roots α_i, and we have $\langle \alpha_i, \alpha_j \rangle = a_{ij} = 2\delta_{i,j} - \delta_{i,j-1} - \delta_{i-1,j}$ (subscripts mod n) where $A = (a_{ij})$ is the Cartan matrix of $\mathfrak{a}_n$. The whole algebra is generated by $\{x_i, y_i : i = 0, \ldots, n\}$, where $x_i = E_{i,i+1} \in L^{\alpha_i}$, $y_i = E_{i+1,i} \in L^{-\alpha_i}$, $\alpha_0 = -(\alpha_1 + \cdots + \alpha_n)$, and the commutator structure satisfies $[h_i, x_j] = a_{ij} x_j$, $[h_i, y_j] = -a_{ij} y_j$ and $[x_i, y_j] = \delta_{ij} y_j$.

First, consider $\mathfrak{a}_2$. We know that $W(\mathfrak{a}_2) \cong S_3$ and there are three different conjugacy classes corresponding to the three possible partitions: $3 = 2+1 = 1+1+1$. For $\mathcal{P} = \{1,1,1\}$, the associated regular subalgebra is $\mathfrak{a}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_0 = \{0\}$, so the Carter diagram is $\emptyset$ and the resulting conjugacy class is $\{\text{id}\}$. It follows that the CSA decomposes as $\mathfrak{h} = \mathfrak{h}'_2$ in the notation hitherto adopted, as all of $\mathfrak{h}$ is fixed by $w = \text{id}$. As is easily seen, for all algebras $\mathfrak{a}_n$, the partition $\mathcal{P} = \{1, \ldots, 1\}$ is associated with $w = \text{id}$. This class generates the so-called homogeneous Heisenberg subalgebra, and regardless of the Lie algebra, all the eigenvectors are regular, as the only eigenspace is $\mathfrak{h}$ itself, and all CSA's are maximal Abelian.

For $\mathcal{P} = \{2, 1\}$, the regular subalgebra is $\mathfrak{a}_1$ and the Carter diagram is $\circ$. The conjugacy class of $W(\mathfrak{a}_2)$ in this case is $[r_{\alpha_1}]$, since α_1 forms the simple root of a subalgebra Sp $\{h_1, x_1, y_1\} \cong \mathfrak{a}_1$ and r_{α_1} is the (primitive) Coxeter element. In terms of S_3, α_1 is the transposition $(1\ 2)$ so that the conjugacy class consists of the three possible 2-cycles. Passing back to $W(\mathfrak{a}_2)$, this means $[r_{\alpha_1}] = \{r_{\alpha_0}, r_{\alpha_1}, r_{\alpha_2}\}$, merely reflecting the fact that the resulting subalgebras Sp $\{h_i, x_i, y_i\}$, $i = 0, 1, 2$, are all isomorphic to $\mathfrak{a}_1$. Since the partition is of the form $\{p, \ldots, p, 1\}$, the conjugacy class admits regular eigenvectors and the CSA decomposes as $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_1'$. In the case of the representative r_{α_1}, we have $\mathfrak{h}_1 = \text{Sp}\ \{h_1\}$ and $\mathfrak{h}_1' = \text{Sp}\ \{h_1 + 2h_2\}$.

The remaining partition is $\mathcal{P} = \{3\}$, for which the regular subalgebra is $\mathfrak{a}_2$ itself, with Carter diagram $\circ - \circ$. The conjugacy class is represented by the Coxeter element of $\mathfrak{a}_2$, $w_C = r_{\alpha_1} r_{\alpha_2}$, and the entire class is just $\{r_{\alpha_1} r_{\alpha_2}, r_{\alpha_2} r_{\alpha_1}\}$ (this is the class of 3-cycles in S_3). Again, the class admits regular eigenvectors (this is always the case with Coxeter classes) and the CSA decomposes as $\mathfrak{h} = \mathfrak{h}_2$ and $\mathfrak{h}_2' = \{0\}$ (the Coxeter element has no fixed points). As mentioned earlier, this class produces the principal Heisenberg subalgebra.

Next, consider $\mathfrak{a}_3$. Now there are five different conjugacy classes corresponding to the partitions of 4: $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$. $\mathcal{P} = \{1, 1, 1, 1\}$ gives $w = \text{id}$, with Carter diagram $\emptyset$ and CSA decomposition $\mathfrak{h} = \mathfrak{h}_3'$. At the other end of the scale, $\mathcal{P} = \{4\}$ corresponds to $\mathfrak{a}_3$ itself, with Carter diagram $\circ - \circ - \circ$, and representative $w_C = r_{\alpha_1} r_{\alpha_2} r_{\alpha_3}$. The conjugacy class corresponds to the six 4-cycles of S_4.

The partition $\mathcal{P} = \{2, 1, 1\}$ yields the regular subalgebra $\mathfrak{a}_1$, with conjugacy class $[r_{\alpha_1}]$ analogous to the six 2-cycles of S_4. The CSA decomposition for the representative r_{α_1} is $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2' = \text{Sp}\ \{h_1\} \oplus \text{Sp}\ \{h_1 + 2h_2, h_3\}$. Note that $\mathcal{P}$ is neither of the form $\{p, \ldots, p\}$ nor $\{p, \ldots, p, 1\}$, hence the representatives admit no regular eigenvectors.

For $\mathcal{P} = \{2, 2\}$, the regular subalgebra is $\mathfrak{a}_1 \oplus \mathfrak{a}_1$, with Carter diagram $\circ\ \circ$. Since the nodes are disconnected, the corresponding roots must be orthogonal. One particular choice is $w = r_{\alpha_1} r_{\alpha_3}$, which is a product of disjoint 2-cycles, $(1\ 2)(3\ 0)$, in the S_4 picture. The conjugacy class consists of the three possible products of disjoint 2-cycles. It admits regular eigenvectors and, for the chosen representative, the CSA decomposes as $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_1' = \text{Sp}\ \{h_1\} \oplus \text{Sp}\ \{h_3\} \oplus \text{Sp}\ \{h_1 + 2h_2 + h_3\}$.

Finally, for $\mathcal{P} = \{3,1\}$, the regular subalgebra is $\mathfrak{a}_2$ with Carter diagram $\circ - \circ$. A representative of the conjugacy class is therefore $w = r_{\alpha_1} r_{\alpha_2}$ and the corresponding class in S_4 is that consisting of the eight 3-cycles. It admits regular eigenvectors and, for the chosen representative, the CSA decomposes as $\mathfrak{h} = \mathfrak{h}_2 \oplus \mathfrak{h}_1' = \mathrm{Sp}\ \{h_1, h_2\} \oplus \mathrm{Sp}\ \{h_1 + 2h_2 + 3h_3\}$.

3.2 The Lift of the Weyl Group

In order to generate hierarchies of partial differential equations associated with a particular Heisenberg subalgebra of $\mathfrak{g}$, it becomes necessary to lift the appropriate Weyl group element, which may be regarded as an automorphism of $\mathfrak{h}$, to an automorphism on the whole algebra, $\mathfrak{g}$. In other words, we need to find an automorphism of $\mathfrak{g}$ whose restriction to $\mathfrak{h}$ gives back the original Weyl group element. For the reflections, r_α, where α is a root, there is a natural lift [23, §33], $\tilde{r}_\alpha$, to $\mathrm{Int}(\mathfrak{g})$, the subgroup of inner automorphisms of $\mathrm{Aut}(\mathfrak{g})$, given by

$$\tilde{r}_\alpha = \mathrm{Ad}\ \exp\ \left[i\pi(2\langle\alpha,\alpha\rangle)^{-\frac{1}{2}}(E_\alpha + E_{-\alpha})\right], \tag{3.1}$$

where $E_{\pm\alpha} \in L^{\pm\alpha}$ such that $[E_\alpha, E_{-\alpha}] = h_\alpha$, corresponding to α via the Killing form. In particular, for the simple roots, $\langle\alpha_i,\alpha_i\rangle = 2$ for all the simple Lie algebras, as does $\langle\alpha_0,\alpha_0\rangle$, where $-\alpha_0$ is the highest positive root, so that

$$\tilde{r}_{\alpha_i} = \mathrm{Ad}\ \exp\ \left[\tfrac{i\pi}{2}(E_{\alpha_i} + E_{-\alpha_i})\right],\ \forall i = 0, 1, \ldots, \mathrm{rank}\ \mathfrak{g}. \tag{3.2}$$

In the case of algebras of type $\mathfrak{a}$, E_{α_i} corresponds to x_i and $E_{-\alpha_i}$ to y_i, in the notation previously adopted. Note that $\tilde{r}_\alpha\ |_{\mathfrak{h}} = r_\alpha$ and $\tilde{r}_\alpha$ interchanges E_α and $E_{-\alpha}$. We denote the group generated by the inner automorphisms, $\tilde{r}_{\alpha_i}$, by $\tilde{W}(\mathfrak{g})$. Thus, $\tilde{W}(\mathfrak{g})$ is a subgroup of the inner automorphisms of $\mathfrak{g}$.

The lift, $\tilde{w}$, of an element $w \in W(\mathfrak{g})$, has the property that $\tilde{w}\ |_{\mathfrak{h}} = w$ and $\tilde{w}(L^\alpha) = L^{w(\alpha)}$. Insofar as this is all we require of a lift, it is clear that the prescription for $\tilde{r}_\alpha$ given in Equation (3.1) is not the only one that will work. Helgason [28, §X.4] gives another for simple reflections, namely $\tilde{r}_{\alpha_i} := \mathrm{Ad}\ (\exp x_i \exp(-y_i) \exp x_i)$. In fact, Schellekens and Warner [50] show that the lifts are unique only up to conjugation by elements of $\exp(\mathfrak{h})$, the so-called torus in G, the Lie group of $\mathfrak{g}$. In other words,

$$\tilde{r}_\alpha = \mathrm{Ad}\ \left[\exp\left(i\pi(2\langle\alpha,\alpha\rangle)^{-\frac{1}{2}}(E_\alpha + E_{-\alpha})\right) \exp H\right]$$

will also serve as a lift, where $H \in \mathfrak{h}$ is arbitrary. Schellekens and Warner refer to lifts of the form in Equation (3.1) as *canonical* or *shiftless.* It may be shown [50] that the order of the lift $\tilde{w}$ is either equal to or twice that of $w \in W(\mathfrak{g})$.[1]

For $\mathfrak{a}_n$, Table 4 of [14, §4.1] shows that, for those w admitting a regular eigenvector, the order of w doubles upon lifting if and only if w belongs to the conjugacy class corresponding to the partition $\{p, \ldots, p, 1\}$ of $n+1$, where p is *even.* In particular, the order of the Coxeter element is preserved on lifting, a fact that is true for all the simple Lie algebras.

If the lift $\tilde{w}$ has order N, then $\mathfrak{g}$ may be decomposed into the eigenspaces of $\tilde{w}$:

$$\mathfrak{g} = \bigoplus_{j=1}^{N} \mathfrak{g}_j,$$

where $\tilde{w}(X) = \exp(\frac{2\pi i j}{N})X$ for $X \in \mathfrak{g}_j$. Restricting to the CSA, $\mathfrak{h}$, and recalling that $\tilde{w}\mid_{\mathfrak{h}} = w$, we have the decomposition

$$\mathfrak{h} = \bigoplus_{j \in I[w]} \mathfrak{h}_j,$$

where

$$I[w] = \left\{k \in \mathbb{Z} : 0 \leq k \leq N, \text{eigenvalues of } w \text{ are } \exp\left(\tfrac{2\pi i k}{N}\right)\right\},$$

to borrow the nomenclature of [13]. The set $I[w]$ has cardinality $\mathrm{rank}(\mathfrak{g})$. It will be important in the construction of the Heisenberg subalgebra associated with $[w]$.

[1]It should be mentioned that lifts may be constructed via some other form of group action of G on $\mathfrak{g}$ and that the order of the lift depends on the actual group action in question [50]. For our purposes, it will be enough to confine our attention to the adjoint representation of G.

Example 3.2.1 *The lifts of the conjugacy classes of the Weyl group for $\mathfrak{a}_2$ and $\mathfrak{a}_3$.*

The conjugacy classes have already been described in Example 3.1.1. In all cases, the identity (on $\mathfrak{h}$) lifts to the identity (on $\mathfrak{g}$). For $\mathfrak{a}_2$, $\mathcal{P} = \{2,1\}$ corresponds to $[r_{\alpha_1}]$. From Equation (3.2), we know that

$$\begin{aligned}\tilde{r}_{\alpha_1} &= \text{Ad}\exp\tfrac{i\pi}{2}(x_1+y_1)\\ &= \text{Ad}\exp\begin{pmatrix} 0 & \frac{i\pi}{2} & 0\\ \frac{i\pi}{2} & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \text{Ad}\begin{pmatrix} 0 & i & 0\\ i & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.\end{aligned}$$

The order of $\tilde{r}_{\alpha_1}$ is 4, twice that of r_{α_1}, as predicted by [14]. For later use, it is convenient to describe the action of $\tilde{r}_{\alpha_1}$ in terms of its eigenspaces, as in Table 3.1. The action of r_{α_1} itself may be recovered merely by looking at the eigenvectors lying

Table 3.1: The lift of r_{α_1} for $\mathfrak{a}_2$

Eigenvalue	Eigenvectors
1	$h_1+2h_2 = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}$, $x_1+y_1 = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$
i	$x_2+y_0 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$, $x_0-y_2 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & -1 & 0\end{pmatrix}$
-1	$h_1 = \begin{pmatrix} 1 & 0 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $x_1-y_1 = \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$
$-i$	$x_2-y_0 = \begin{pmatrix} 0 & 0 & -1\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$, $x_0+y_2 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 1 & 0\end{pmatrix}$

in $\mathfrak{h}$. These correspond to the eigenvalues $i^0 = 1$ and $i^2 = -1$, so that $I[r_{\alpha_1}] = \{0, 2\}$.

The Coxeter class has representative $w_C = r_{\alpha_1} r_{\alpha_2}$, which has order 3. The lift may be shown to be

$$\begin{aligned} \tilde{w}_C &= \mathrm{Ad}\ \exp \tfrac{2\pi}{3\sqrt{3}}(y_0 + y_1 + y_2 - x_0 - x_1 - x_2) \\ &= \mathrm{Ad} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

In fact, for the Coxeter element, $w_C = \prod_{i=1}^{n} r_{\alpha_i}$, of $\mathfrak{a}_n$, the lift is given by $\tilde{w}_C = \mathrm{Ad} \sum_{i=0}^{n} y_i$. This is conjugate to the more usual form of the Coxeter automorphism, which is given by Ad diag $(\omega^n, \ldots, \omega, 1)$, where $\omega = \exp(\frac{2\pi i}{n+1})$, about which more will be said later. For completeness, the following is the precise meaning of the term Coxeter automorphism, as given in [18]:

Definition 3.2.2 *An automorphism $C \in Aut(\mathfrak{g})$ is called a Coxeter automorphism if*
(i) the set of fixed points of C is Abelian;
(ii) of all automorphisms, $C\psi$, where $\psi \in Int(\mathfrak{g})$, whose set of fixed points is Abelian, C has least order.

For $\mathfrak{a}_n$, there are two different types of Coxeter automorphism up to conjugation by inner automorphisms. This is due to the fact that $\mathrm{Aut}(\mathfrak{a}_n)/\mathrm{Int}(\mathfrak{a}_n) \cong \mathbb{Z}_2$, the group of automorphisms of the Dynkin diagram. Presently under consideration is the identity coset, $\mathrm{Int}(\mathfrak{g})$. The other type of Coxeter automorphism is connected with the twisted algebra over $\mathfrak{a}_n$.

Returning to $\tilde{w}_C$, its action is given in terms of its eigenspaces in Table 3.2. In this case, the order is preserved upon lifting. In the table, $\omega = \exp(\frac{2\pi i}{3})$. Again, the action of w_C itself is found by looking at those eigenvectors lying in $\mathfrak{h}$, which correspond to the eigenvalues ω and ω^2, so that $I[w_C] = \{1, 2\}$.

Table 3.2: The lift of w_C for $\mathfrak{a}_2$

Eigenvalue	Eigenvectors
1	$x_0 + x_1 + x_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $y_0 + y_1 + y_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
ω	$h_1 - \omega h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$ $x_0 + \omega^2 x_1 + \omega x_2 = \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix}$ $y_0 + \omega^2 y_1 + \omega y_2 = \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}$
ω^2	$h_1 - \omega^2 h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$ $x_0 + \omega x_1 + \omega^2 x_2 = \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix}$ $y_0 + \omega y_1 + \omega^2 y_2 = \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}$

Next, consider $\mathfrak{a}_3$. Example 3.1.1 enumerates all the conjugacy classes. The first class to consider is $[r_{\alpha_1}]$. The lift is

$$\tilde{r}_{\alpha_1} = \operatorname{Ad} \exp \tfrac{i\pi}{2}(x_1 + y_1) = \operatorname{Ad} \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The order of $\tilde{r}_{\alpha_1}$ is 4, and its action is summarised in Table 3.3. Note that

Table 3.3: The lift of r_{α_1} for $\mathfrak{a}_3$

Eigenvalue	Eigenvectors
1	$h_3,\ h_1 + 2h_2,\ x_3,\ y_3,\ x_1 + y_1$
i	$x_2 + [x_1, x_2],\ x_0 - [y_3, y_2],\ y_2 - [y_2, y_1],\ y_0 + [x_2, x_3]$
-1	$h_1,\ x_1 - y_1$
$-i$	$x_2 - [x_1, x_2],\ x_0 + [y_3, y_2],\ y_2 + [y_2, y_1],\ y_0 - [x_2, x_3]$

$$\begin{aligned} [x_1, x_2] &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & [x_2, x_3] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ [y_2, y_1] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & [y_3, y_2] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

We have $I[r_{\alpha_1}] = \{0, 2\}$.

The next class is that containing $w = r_{\alpha_1} r_{\alpha_3}$, corresponding to the regular subalgebra consisting of two copies of $\mathfrak{a}_1$. The lift is given by

$$\begin{aligned} \tilde{w} = \tilde{r}_{\alpha_1}\tilde{r}_{\alpha_3} &= \text{Ad}\exp\tfrac{i\pi}{2}(x_1+y_1+x_3+y_3) \\ &= \text{Ad}\begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \end{aligned}$$

since α_1 and α_3 are orthogonal so that x_1+y_1 and x_3+y_3 commute. Table 3.4 describes the situation in this case. Here, the lift has order 2 and $I[w]=\{0,1\}$.

Table 3.4: The lift of $r_{\alpha_1}r_{\alpha_3}$ for $\mathfrak{a}_3$

Eigenvalue	Eigenvectors
1	$h_1+2h_2+h_3,\ x_1+y_1,\ x_3+y_3,$ $x_2+y_0,\ x_0+y_2,\ [x_1,x_2]+[x_2,x_3],\ [y_2,y_1]+[y_3,y_2]$
-1	$h_1,\ h_3,\ x_1-y_1,\ x_3-y_3,$ $x_2-y_0,\ x_0-y_2,\ [x_1,x_2]-[x_2,x_3],\ [y_2,y_1]-[y_3,y_2]$

The class containing $w=r_{\alpha_1}r_{\alpha_2}$ corresponds to the regular subalgebra, $\mathfrak{a}_2$, for which both the Weyl element and its lift have order 3. See Table 3.5, in which $\omega=\exp(\frac{2\pi i}{3})$, for details. Here $I[w]=\{0,1,2\}$. In this case, the lift is given by

$$\begin{aligned} \tilde{w} &= \text{Ad}\exp\tfrac{2\pi}{3\sqrt{3}}(y_1+y_2+[x_1,x_2]-x_1-x_2-[y_2,y_1]) \\ &= \text{Ad}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Finally, the Coxeter class contains $w_C=r_{\alpha_1}r_{\alpha_2}r_{\alpha_3}$ which has lift of order 4 given

Table 3.5: The lift of $r_{\alpha_1} r_{\alpha_2}$ for $\mathfrak{a}_3$

Eigenvalue	Eigenvectors
1	$h_1 + 2h_2 + 3h_3,\ x_1 + x_2 + [y_2, y_1],\ y_1 + y_2 + [x_1, x_2],$ $x_3 + y_0 + [x_2, x_3],\ y_3 + x_0 + [y_3, y_2]$
ω	$h_1 - \omega h_2,\ x_1 + \omega^2 x_2 + \omega[y_2, y_1],\ y_1 + \omega^2 y_2 + \omega[x_1, x_2],$ $x_3 + \omega^2 y_0 + \omega[x_2, x_3],\ y_3 + \omega^2 x_0 + \omega[y_3, y_2]$
ω^2	$h_1 - \omega^2 h_2,\ x_1 + \omega x_2 + \omega^2[y_2, y_1],\ y_1 + \omega y_2 + \omega^2[x_1, x_2],$ $x_3 + \omega y_0 + \omega^2[x_2, x_3],\ y_3 + \omega x_0 + \omega^2[y_3, y_2]$

by

$$\tilde{w}_C = \mathrm{Ad}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In this case, $I[w_C] = \{1, 2, 3\}$ and Table 3.6 exhibits the action of the lift.

Table 3.6: The lift of w_C for $\mathfrak{a}_3$

Eigenvalue	Eigenvectors
1	$x_0 + x_1 + x_2 + x_3,\ y_0 + y_1 + y_2 + y_3,$ $[x_1, x_2] + [x_2, x_3] + [y_2, y_1] + [y_3, y_2]$
i	$h_1 + (1 - i)h_2 - ih_3,\ x_0 - ix_1 - x_2 + ix_3,\ y_0 - iy_1 - y_2 + iy_3,$ $[x_1, x_2] - i[x_2, x_3] - [y_2, y_1] + i[y_3, y_2]$
-1	$h_1 + h_3,\ x_0 - x_1 + x_2 - x_3,\ y_0 - y_1 + y_2 - y_3,$ $[x_1, x_2] - [x_2, x_3] + [y_2, y_1] - [y_3, y_2]$
$-i$	$h_1 + (1 + i)h_2 + ih_3,\ x_0 + ix_1 - x_2 - ix_3,\ y_0 + iy_1 - y_2 - iy_3,$ $[x_1, x_2] + i[x_2, x_3] - [y_2, y_1] - i[y_3, y_2]$

3.3 $\tilde{W}(\mathfrak{g})$ and Kač's Classification of Aut($\mathfrak{g}$)

The classification of automorphisms of finite order of simple, finite dimensional Lie algebras was originally achieved by Kač [35] and subsequently recapitulated by Helgason [28]. The main body of this work is contained in Theorems X 5.15 and X 5.16 of [28], which are reproduced here:

Theorem 3.3.1 (Kač, Helgason) *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, ν a fixed automorphism of $\mathfrak{g}$ of order k ($k = 1, 2, 3$) induced by an automorphism of the Dynkin diagram for a CSA $\tilde{\mathfrak{h}}$ of $\mathfrak{g}$. Let $\mathfrak{g} = \oplus_j \mathfrak{g}_j^\nu$ be the corresponding $\mathbb{Z}_k$-gradation of eigenspaces. The fixed point set $\mathfrak{h}^\nu$ of ν in $\tilde{\mathfrak{h}}$ is a CSA of the (simple) Lie algebra $\mathfrak{g}_0^\nu$. Fix canonical generators x_j, y_j, h_j ($1 \leq j \leq n$) of $\mathfrak{g}_0^\nu$ corresponding to the simple roots $\alpha_1, \ldots, \alpha_n$ in $\Delta(\mathfrak{g}_0^\nu, \mathfrak{h}^\nu)$. Let $\tilde{\alpha}_0$ be the lowest root of $L(\mathfrak{g}, \nu)$ of the form $(\alpha_0, 1)$ and fix $x_0 \neq 0$ in $\mathfrak{g}_1^\nu$ such that $\lambda x_0 \in L(\mathfrak{g}_0, \nu)^{\tilde{\alpha}_0}$. Let $(s_0, s_1, \ldots, s_n)$ be nonnegative relatively prime integers and put $N = k \sum_0^n a_j s_j$ where $(a_0, \ldots, a_n)$ is a left zero eigenvector of the Cartan matrix of $\mathfrak{g}^{(k)}$, normalized so that $a_0 = 1$. Let $\epsilon = \exp(\frac{2\pi i}{N})$. Then:*

(i) The vectors $x_0, x_1, \ldots, x_n$ generate $\mathfrak{g}$ and the relations

$$\sigma_{\mathbf{s},k}(x_j) = \epsilon^{s_j} x_j, \;\; 0 \leq j \leq n$$

uniquely define an automorphism of $\mathfrak{g}$ of order N, to be called an automorphism of type $(s_0, \ldots, s_n; k)$.

(ii) Let $j_1, \ldots, j_t$ be all the indices for which $s_{j_1} = \cdots = s_{j_t} = 0$. Then $\mathfrak{g}_0$, the set of fixed points of σ, is the direct sum of an $(n-t)$-dimensional centre and a semisimple Lie algebra whose Dynkin diagram is the subdiagram of the diagram of $\mathfrak{g}^{(k)}$ consisting of the vertices $j_1, \ldots, j_t$.

(iii) The automorphisms $\sigma_{\mathbf{s},k}$ exhaust all N-th order automorphisms, up to conjugation.

(iv) $\sigma_{\mathbf{s},k}$ is an inner automorphism if and only if $k = 1$. Furthermore, k is the smallest positive integer for which $(\sigma_{\mathbf{s},k})^k$ is inner.

(v) $\sigma_{\mathbf{s},k}$ and $\sigma_{\mathbf{s}',k'}$ are conjugate if and only if $k = k'$ and $\mathbf{s}$ can be transformed into $\mathbf{s}'$ by an automorphism of the Dynkin diagram of $\mathfrak{g}^{(k)}$.

A few comments are in order:

(i) $\mathfrak{g}_j^\nu$ are the eigenspaces of ν associated with the eigenvalue ω^j, where $\omega = \exp(\frac{2\pi i}{k})$.

(ii) $k = 3$ only occurs for $\mathfrak{g} = \mathfrak{d}_4$.

(iii) n is the rank of $\mathfrak{g}_0^\nu$.

(iv) From the action of $\sigma_{\mathbf{s},k}$ on the x_j, it follows that $\sigma_{\mathbf{s},k}(y_j) = \epsilon^{-s_j} y_j$ and that $\sigma_{\mathbf{s},k}(h_j) = h_j$ for $j = 0, \ldots, n$.

(v) When $k = 1$, ν is just the identity automorphism and $\mathfrak{g}_0^\nu$ is just $\mathfrak{g}$ itself, while $\mathfrak{h}^\nu = \tilde{\mathfrak{h}}$.

(vi) The identity automorphism is of type $(\mathbf{s}; 1)$ where $\mathbf{s} = (1, 0, \ldots, 0)$.

In light of this, we see that the lifts of the representatives of the conjugacy classes of the Weyl group, which are all inner, are conjugate in $\mathrm{Aut}(\mathfrak{g})$ to automorphisms of the form $\sigma_{\mathbf{s}} \equiv \sigma_{\mathbf{s},1}$ of type $(s_0, \ldots, s_n; 1)$ and hence define a gradation on $\mathfrak{g}^{(1)}$. In order to find the vector $\mathbf{s}$ associated to a given conjugacy class of $W(\mathfrak{g})$, it is useful to introduce the notion of a *shift vector* [3]. Given $\mathbf{s}$, we define a shift vector $\gamma_{\mathbf{s}} \in \mathfrak{h}^*$ by

$$\langle \gamma_{\mathbf{s}}, \alpha_k \rangle = \frac{s_k}{N}, \; k = 1, \ldots, n,$$

where $N = \sum_{j=0}^{n} a_j s_j$. This is equivalent to saying that

$$\gamma_{\mathbf{s}} = \frac{1}{N} \sum_{j=1}^{n} \frac{2}{\langle \alpha_j, \alpha_j \rangle} s_j \lambda_j,$$

where the λ_j, $j = 1, \ldots, n$, are the *fundamental dominant weights* with respect to the simple roots, α_j, $j = 1, \ldots, n$ [33]. These form a basis of $\mathfrak{h}^*$ dual to the basis $\{2\alpha_j / \langle \alpha_j, \alpha_j \rangle : j = 1, \ldots, n\}$ with respect to the inner product, and so

$$2 \frac{\langle \lambda_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}.$$

The automorphism $\sigma_{\mathbf{s}} \equiv \sigma_{\mathbf{s},1}$ of order N is now described by

$$\sigma_{\mathbf{s}}(x_j) = \exp(2\pi i \langle \gamma_{\mathbf{s}}, \alpha_j \rangle) x_j,$$

where $x_j \in L^{\alpha_j}$, $i = 0, \ldots, n$, from which it follows that

$$\sigma_s(x_\alpha) = \exp(2\pi i \langle \gamma_s, \alpha \rangle) x_\alpha, \tag{3.3}$$

whenever $x_\alpha \in L^\alpha$. Note that this includes the case when $\alpha = 0$ as then $L^\alpha = \mathfrak{h}$.

Each α_i is a linear combination of the λ_j, say $\alpha_i = \sum_{j=1}^n m_{ij}\lambda_j$, so that

$$a_{ik} = 2\frac{\langle \alpha_i, \alpha_k \rangle}{\langle \alpha_k, \alpha_k \rangle} = \sum_j m_{ij} \cdot 2\frac{\langle \lambda_j, \alpha_k \rangle}{\langle \alpha_k, \alpha_k \rangle} = \sum_j m_{ij}\delta_{jk} = m_{ik},$$

where $A = (a_{ik})$ is the Cartan matrix of $\mathfrak{g}$. It follows that the Cartan matrix represents the change of basis, so that explicitly calculating the λ_i in terms of the α_j requires inverting this (nonsingular) matrix. In the case of $\mathfrak{a}_n$,

$$\gamma_s = \frac{1}{N}\sum_{j=1}^n s_j \lambda_j,$$

where [33]

$$\lambda_j = \frac{1}{n+1}\left[(n-j+1)\sum_{k=1}^j k\alpha_k + j\sum_{k=j+1}^n (n-k+1)\alpha_k\right].$$

In particular, the fundamental dominant weights are listed for $\mathfrak{a}_n$, $n = 1, 2, 3$:

$$\begin{aligned}
\mathfrak{a}_1 &: \lambda_1 = \tfrac{1}{2}\alpha \\
\mathfrak{a}_2 &: \lambda_1 = \tfrac{1}{3}(2\alpha_1 + \alpha_2),\ \lambda_2 = \tfrac{1}{3}(\alpha_1 + 2\alpha_2) \\
\mathfrak{a}_3 &: \lambda_1 = \tfrac{1}{4}(3\alpha_1 + 2\alpha_2 + \alpha_3),\ \lambda_2 = \tfrac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3), \\
&\quad \lambda_3 = \tfrac{1}{4}(\alpha_1 + 2\alpha_2 + 3\alpha_3)
\end{aligned}$$

It has already been seen that the conjugacy classes of $W(\mathfrak{g})$ may be represented by primitive elements in the semisimple regular subalgebras. For $\mathfrak{a}_n$, the regular subalgebras consist of direct sums of $\mathfrak{a}_m$ where $m \leq n$, for which the only primitive class is the Coxeter class. The shift vector associated to the Coxeter class in any simple algebra of rank n has long been known and goes back to a result due to Kostant [38]. This says that the lift, $\tilde{w}_C$, of the Coxeter element is conjugate to the automorphism of type $(\mathrm{s}_C; 1)$ where $\mathrm{s}_C = (1, \ldots, 1)$ ($n+1$ entries). Hence, the shift vector for the Coxeter element is

$$\gamma_{\mathrm{s}_C} = \frac{1}{h} \sum_{j=1}^{n} \frac{2}{\langle \alpha_j, \alpha_j \rangle} \lambda_j,$$

where h is the Coxeter number of $\mathfrak{g}$ (h equals the order of w_C). For algebras such as $\mathfrak{a}_n$, where $\langle \alpha_j, \alpha_j \rangle = 2 \; \forall j$,

$$\gamma_{\mathrm{s}_C} = \frac{1}{h} \rho,$$

where it may be shown [33] that

$$\rho = \sum_j \lambda_j = \tfrac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

In particular, for $\mathfrak{a}_n$,

$$\gamma_{\mathrm{s}_C} = \frac{1}{2(n+1)} \sum_{\alpha \in \Delta^+} \alpha,$$

and again, for later use, the specific cases of $\mathfrak{a}_n$, $n = 1, 2, 3$, are calculated:

$$\begin{array}{lllll}
\mathfrak{a}_1 & : & \gamma_{(1,1)} & = & \frac{1}{4}\alpha \\
\mathfrak{a}_2 & : & \gamma_{(1,1,1)} & = & \frac{1}{6}(\alpha_1 + \alpha_2 + (\alpha_1 + \alpha_2)) = \frac{1}{3}(\alpha_1 + \alpha_2) \\
\mathfrak{a}_3 & : & \gamma_{(1,1,1,1)} & = & \frac{1}{8}(\alpha_1 + \alpha_2 + \alpha_3 + (\alpha_1 + \alpha_2) + (\alpha_2 + \alpha_3) + (\alpha_1 + \alpha_2 + \alpha_3)) \\
 & & & = & \frac{1}{8}(3\alpha_1 + 4\alpha_2 + 3\alpha_3)
\end{array}$$

As may be easily verified from the previous expressions for the fundamental dominant weights for these algebras, the same results are obtained from taking $\rho = \sum_j \lambda_j$.

In general, to find the shift vector for a given conjugacy class of $W(\mathfrak{g})$, we choose a representative $w = w_1 \ldots w_r$, where each w_j is a primitive element in the simple constituent, $\mathfrak{g}_j$, of the regular subalgebra $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$ of $\mathfrak{g}$, and embed the shift vector $\gamma = \gamma_1 + \cdots + \gamma_r$ into the form

$$\gamma = \sum_{k=1}^{n} c_k \alpha_k,$$

as a linear combination of the simple roots of $\mathfrak{g}$. Here γ_j denotes the primitive shift vector of the simple subalgebra $\mathfrak{g}_j$ corresponding to w_j. These may be added to give γ, since the primitive shift vectors of the $\mathfrak{g}_j$ are mutually orthogonal. We then equate this expression for γ to the general form for a shift vector in $\mathfrak{g}$,

$$\begin{aligned} \gamma &= \frac{1}{N}\sum_{j=1}^{n} s_j\lambda_j \\ &= \frac{1}{N}\sum_{k=1}^{n}\left(\sum_{j=1}^{n} s_j b_{jk}\right)\alpha_k, \end{aligned}$$

where N is the order of the lift, $\tilde{w}$, and solve the resulting linear system of equations in s_j. As the b_{jk} are just the entries of the inverse of the Cartan matrix, A, it follows that

$$(s_1, \ldots, s_n) = N \cdot \mathbf{c}A,$$

where $\mathbf{c} = (c_1, \ldots, c_n)$. We have therefore solved the unique solution for $\mathbf{s}$ on observing that

$$s_0 = N - \sum_{j=1}^{n} a_j s_j.$$

Thus, the problem reduces to that of finding the shift vectors for the primitive elements (of which the Coxeter elements are a subset) of the simple Lie algebras. For $\mathfrak{a}_n$, there is nothing more to do, as the Coxeter elements are the only primitive elements. More generally, the primitive shift vectors may be determined by a process of exhaustion: the trace formula in [32] is often enough, but Bouwknegt [3] gives another three criteria, namely quasirationality, mass shift and defect (the last of which is also discussed in [32] in ascertaining the conjugacy classes of $W(\mathfrak{e}_8)$) which, together with the aforementioned trace formula, will ensure the determination of any primitive shift vector.

REMARK: Having embedded the shift vector into $\mathfrak{g}$ and solved for the s_j, the resulting vector $\mathbf{s}$ is not guaranteed to have exclusively nonnegative entries. Nonetheless, it still represents an automorphism $\sigma_{\mathbf{s}}$, defined in exactly the same way as in Theorem 3.3.1, that is an automorphism of order $N = \sum_0^n a_j s_j$ generated by requiring $\sigma_{\mathbf{s}}(X_j) = \epsilon^{s_j} X_j$, for $j = 0, \ldots, n$. Furthermore, the vector $\mathbf{s}$ still generates a gradation of $\mathfrak{g}^{(1)}$ in exactly the same manner described in §2.5. Moreover, $\sigma_{\mathbf{s}}$ is conjugate to the lift of the representative $w = w_1 \ldots w_r$ orginally chosen. In order to obtain an automorphism of the actual form of Theorem 3.3.1, that is one whose representing vector $\mathbf{s}$ consists of entirely nonnegative entries, we can follow a procedure outlined in [3] and attributed therein to M. Bos. Given the $(n+1)$-tuple $\mathbf{s} = (s_0, s_1, \ldots, s_n)$ of integers, at least one of which is negative, say s_{j_0}, $j_0 \neq 0$, with associated automorphism $\sigma_{\mathbf{s}}$, consider the (conjugate) automorphism $\tilde{r}_{\alpha_{j_0}} \sigma_{\mathbf{s}} \tilde{r}_{\alpha_{j_0}}^{-1}$. It is not hard to show that this is $\sigma_{\mathbf{s}'}$, where $s_j' = s_j - a_{jj_0} s_{j_0}$, $j = 1, \ldots, n$, and s_0' is chosen to preserve the order, so that $\sum_{j=0}^n a_j (s_j' - s_j) = 0$. (Alternatively, we could describe $\mathbf{s}'$ as being given by $s_j' = s_j - a_{jj_0} s_{j_0}$, $j = 0, \ldots, n$, where $(a_{jj_0})_{j=0}^n$ is the j_0th column of the extended Cartan matrix of the affine algebra $\mathfrak{g}^{(1)}$, since $\sum_{j=0}^n a_j \alpha_j = 0$, α_0 being the negative of the highest root, δ. This approach also subsumes the case when $j_0 = 0$.) We keep repeating this procedure until we arrive at a vector, $\mathbf{s}_+$, containing only nonnegative, relatively prime integers. That this can be achieved in finitely many steps has been established by Bos (see [3]). Thus, we eventually obtain an automorphism $\sigma_{\mathbf{s}_+}$ of the form in Theorem 3.3.1 which is conjugate to our original $\sigma_{\mathbf{s}}$ by a series of lifts of simple reflections. For the purposes of this work, however, it is the vector $\mathbf{s} = (s_0, N \cdot \mathbf{c}A)$ and its associated gradation which is of importance. For clarity, this vector $\mathbf{s}$ shall hereafter be referred to as $\mathbf{s}[w]$, in order to indicate the conjugacy class to which it is associated. □

Example 3.3.2 *The* $\mathbf{s}$*-vectors associated with the conjugacy classes of* $\mathfrak{a}_2$ *and* $\mathfrak{a}_3$*.*

The lifts for each conjugacy class were constructed in Example 3.2.1. For $W(\mathfrak{a}_2)$, there are two nontrivial conjugacy classes, the first of which is $[r_{\alpha_1}]$. The corresponding regular subalgebra is $\mathfrak{a}_1$ with simple root, α_1, and shift vector,

$$\gamma = \gamma_{(1,1)} = \tfrac{1}{4}\alpha_1,$$

so that $\mathbf{c} = (\frac{1}{4}, 0)$. We want to solve

$$\gamma = \frac{1}{N}(s_1\lambda_1 + s_2\lambda_2),$$

where $N = 4$ is the order of the lift, $\tilde{r}_{\alpha_1}$. Thus,

$$\begin{aligned}(s_1, s_2) = 4\mathbf{c}A &= 4\left(\tfrac{1}{4}, 0\right)\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\ &= (2, -1).\end{aligned}$$

Hence, $s_0 = 4 - (2 - 1) = 3$ and $\mathbf{s}[w] = (3, 2, -1)$. For completeness, we find the corresponding $\mathbf{s}_+$-vector. Successively applying the algorithm $s_j' = s_j - a_{jj_0} s_{j_0}$, $j = 0, 1, 2$, results in

$$\mathbf{s}[w] = (3, 2, -1) \xrightarrow{j_0=2} (2, 1, 1).$$

Thus, $\mathbf{s}[w] = (3, 2, -1)$ and $\mathbf{s}_+ = (2, 1, 1)$ for $[r_{\alpha_1}]$.

The remaining conjugacy class for $\mathfrak{a}_2$ is the Coxeter class which, as already seen, has $\mathbf{s}[w_C] = (1, 1, 1)$.

For $W(\mathfrak{a}_3)$, the first conjugacy class is $[r_{\alpha_1}]$, with corresponding regular subalgebra $\mathfrak{a}_1$, possessing simple root α_1 and shift vector

$$\gamma = \gamma_{(1,1)} = \tfrac{1}{4}\alpha_1,$$

so that $\mathbf{c} = (\frac{1}{4}, 0, 0)$. As the lift has order 4, we have

$$\begin{aligned}(s_1, s_2, s_3) = 4\mathbf{c}A &= 4\left(\tfrac{1}{4}, 0, 0\right)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \\ &= (2, -1, 0).\end{aligned}$$

Hence, $s_0 = 4 - (2 - 1 + 0) = 3$ whence

$$\mathbf{s}[w] = (3, 2, -1, 0) \xrightarrow{j_0=2} (3, 1, 1, -1) \xrightarrow{j_0=3} (2, 1, 0, 1).$$

Thus, $\mathbf{s}[w] = (3, 2, -1, 0)$ and $\mathbf{s}_+ = (2, 1, 0, 1)$ for $[r_{\alpha_1}]$.

The next conjugacy class is $[r_{\alpha_1} r_{\alpha_3}]$, with corresponding regular subalgebra $\mathfrak{a}_1 \oplus \mathfrak{a}_1$, with simple roots α_1, α_3 for the simple constituents and shift vector

$$\gamma = \gamma_1 + \gamma_2 = \tfrac{1}{4}\alpha_1 + \tfrac{1}{4}\alpha_3,$$

so that $\mathbf{c} = (\frac{1}{4}, 0, \frac{1}{4})$. The lift has order 2, therefore

$$(s_1, s_2, s_3) = 2\mathbf{c}A = (1, -1, 1).$$

Thus, $s_0 = 1$ and

$$\mathrm{s}[w] = (1, 1, -1, 1) \overset{j_0=2}{\longrightarrow} (1, 0, 1, 0),$$

so that $\mathrm{s}[w] = (1, 1, -1, 0)$ and $\mathrm{s}_+ = (1, 0, 1, 0)$ for $[r_{\alpha_1} r_{\alpha_3}]$.

Aside from the Coxeter class for which $\mathrm{s}[w_C] = (1, 1, 1, 1)$, the only other class to consider is $[r_{\alpha_1} r_{\alpha_2}]$, with corresponding regular subalgebra $\mathfrak{a}_2$ having simple roots α_1, α_2, and shift vector

$$\gamma = \gamma_{(1,1,1)} = \tfrac{1}{3}(\alpha_1 + \alpha_2),$$

so that $\mathbf{c} = (\frac{1}{3}, \frac{1}{3}, 0)$. The lift has order 3, therefore

$$(s_1, s_2, s_3) = 3\mathbf{c}A = (1, 1, -1).$$

Thus, $s_0 = 2$ and

$$\mathrm{s}[w] = (2, 1, 1, -1) \overset{j_0=3}{\longrightarrow} (1, 1, 0, 1),$$

so that $\mathrm{s}[w] = (2, 1, 1, -1)$ and $\mathrm{s}_+ = (1, 1, 0, 1)$ for $[r_{\alpha_1} r_{\alpha_2}]$.

3.4 From the Conjugacy Class to the Heisenberg Subalgebra

In the previous section, it was seen how to find an automorphism of the form σ_{s} in Theorem 3.3.1 conjugate to the lift of a given element of a conjugacy class of $W(\mathfrak{g})$. The specific cases for $\mathfrak{a}_2$ and $\mathfrak{a}_3$ were constructed by way of example. Once the conjugating map $\psi \in \mathrm{Aut}(\mathfrak{g})$ is known, for which $\sigma = \psi\tilde{w}\psi^{-1}$, the corresponding Heisenberg subalgebra may be found from $\psi(\mathfrak{h})$, which is itself a CSA. A basis may be found for $\psi(\mathfrak{h})$ in which the elements are of the form $\sum_j z_j$, where the z_j are all of the same $\mathrm{s}[w]$-degree mod N. In fact, the basis of $\mathfrak{h}$ formed by the eigenvectors of w gets mapped to just such a basis of $\psi(\mathfrak{h})$, since ψ takes eigenvectors of w to those of σ, and any eigenspace of σ consists of elements of the same $\mathsf{s}[w]$-degree mod N, as will soon be shown. It is then simply a matter of lifting the z_j into $\mathfrak{g}^{(1)}$ so as to obtain $\sum_j \hat{z}_j$, where the $\hat{z}_j$ are now of the same s-degree. More precisely, if z_j has degree $-k' \equiv k \bmod N$, $k' \in \{0, \ldots, N-1\}$, then z_j lifts to $\lambda^{n+1} z_j$ with degree $Nn+k$, while if z_j has degree k, $k \in \{0, \ldots, N-1\}$, then z_j lifts to $\lambda^n z_j$ with degree $Nn+k$.

Definition 3.4.1 *The subalgebra $\mathcal{H}[w]$ of $\mathfrak{g}^{(1)}$ thus obtained is called the* Heisenberg subalgebra *associated to the conjugacy class $[w]$ of $W(\mathfrak{g})$.*

REMARK: A more formal definition of a Heisenberg subalgebra goes back to [37]: Let $\mathcal{H}$ be the variety of CSA's of $\mathfrak{g}$ and let $\tilde{\mathcal{H}}$ be its loop space, the space of all regular rational maps of $\mathbb{C}^\times$, the extended complex plane, into $\mathcal{H}$. Given a loop $s \in \tilde{\mathcal{H}}$, the associated Heisenberg subalgebra of the loop algebra $\tilde{\mathfrak{g}}$ is defined to be

$$\left\{ \tilde{s} = p \in \tilde{\mathfrak{g}} : p(t) \in s(t), \ \forall t \in \mathbb{C}^\times \right\}.$$

This is a maximal Abelian subalgebra of the loop algebra, $\tilde{\mathfrak{g}}$. Regarding s as a vector bundle over $\mathbb{C}^\times$ with fibre $s(t)$ over $t \in \mathbb{C}^\times$, $\tilde{s}$ may be regarded as a section of the bundle s. When s is the map taking $\mathbb{C}^\times$ to a single point (CSA) $\mathfrak{h} \in \mathcal{H}$, then

$$\tilde{s} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h}$$

is the *homogeneous* Heisenberg subalgebra of $\tilde{\mathfrak{g}}$. When s takes $t \in \mathbb{C}^\times$ to the centralizer in $\mathfrak{g}$ of the element

$$tE_{-\delta} + \sum_{j=1}^{n} E_{\alpha_j},$$

where δ is the highest root and $\{\alpha_j : 1 \leq j \leq n\}$ is the set of simple roots where $E_{-\delta}$, E_{α_j} are elements of the corresponding root spaces, then $\tilde{s}$ is called the *principal* Heisenberg subalgebra of $\tilde{\mathfrak{g}}$. □

A crucial feature of the gradation induced by $\mathsf{s}[w]$ is that the elements of $\mathcal{H}[w]$ are homogeneous with respect to this gradation. In other words, $\mathcal{H}[w]$ has direct sum decomposition

$$\mathcal{H}[w] = \bigoplus_{j\in\mathbb{Z}} \mathcal{H}_j[w],$$

where $\mathcal{H}_j[w]$ consists of elements of degree j and $\mathcal{H}_j[w] = \emptyset$ if j mod $N \notin I[w]$.

Note that the Heisenberg subalgebra associated to the identity element of $W(\mathfrak{g})$ is just $\{\lambda^n \otimes \mathfrak{h} : n \in \mathbb{Z}\}$ and any element $\lambda^n h$, $h \in \mathfrak{h}$, has degree Nn. $\mathcal{H}[\mathrm{id}]$ is called the *homogeneous* Heisenberg subalgebra of $\mathfrak{g}^{(1)}$. Note also that representatives w_1, w_2 of the same conjugacy class yield isomorphic Heisenberg subalgebras.

The gradation given by $\mathbf{s}[w]$ can also be described in the following way. Define a derivation $d_{\mathbf{s}[w]} : \mathfrak{g}^{(1)} \to \mathfrak{g}^{(1)}$ by

$$\begin{aligned} d_{\mathbf{s}[w]} &= d + N\gamma_{\mathbf{s}[w]} \cdot \pi_\Delta \\ &= d + \sum_{j=1}^{n} s_j \lambda_j \cdot \pi_\Delta, \end{aligned}$$

where $d = N\lambda \frac{d}{d\lambda}$ and $\gamma_{\mathbf{s}[w]} = \frac{1}{N}\sum_{j=1}^{n} s_j\lambda_j$ is the shift vector for $\mathbf{s}[w]$. The function $\lambda_j \cdot \pi_\Delta$ is defined by

$$\lambda_j \cdot \pi_\Delta(E_\alpha) = \langle \lambda_j, \alpha \rangle E_\alpha,$$

where $E_\alpha \in L^\alpha$, and extended linearly. If $\alpha = 0$, $L^\alpha = \mathfrak{h}$, so $\lambda_j \cdot \pi_\Delta(H) = 0$, $\forall H \in \mathfrak{h}$. The homogeneous elements of the gradation are then the eigenvectors of $d_{\mathbf{s}[w]}$ with degree equal to the respective eigenvalue. Referring back to (3.3), it follows that for a homogeneous element X with respect to the grading,

$$\sigma(X) = \epsilon^{\deg X} X,$$

where ϵ is a primitive Nth root of unity, thus confirming the earlier assertion that all elements of a given eigenspace of σ have the same $\mathbf{s}[w]$-degree mod N. Note that it is the vector $\mathbf{s}[w] = (s_0, N \cdot \mathbf{c}A)$, which solves the coefficients s_j in the expression for $\gamma_{\mathbf{s}[w]}$, that is meant here, *not* the associated $\mathbf{s}_+$-vector.

At this point we introduce some convenient notation. We define

$$\mathfrak{g}_j^{(1)}(\mathbf{s}[w])$$

to be the subspace of $\mathfrak{g}^{(1)}$ of elements of $\mathrm{s}[w]$-degree j. In a similar vein, for $j \in \mathbb{Z}$, we define

$$\mathfrak{g}^{(1)}_{>j}(\mathrm{s}[w]) := \bigoplus_{l>j} \mathfrak{g}^{(1)}_{l}(\mathrm{s}[w]),$$

the subspace of elements of $\mathrm{s}[w]$-degree $> j$. Similarly, we define

$$\mathfrak{g}^{(1)}_{<j}(\mathrm{s}[w]),\ \mathfrak{g}^{(1)}_{\leq j}(\mathrm{s}[w]),\ \mathfrak{g}^{(1)}_{\geq j}(\mathrm{s}[w]),$$

in the obvious manner.

Example 3.4.2 *The Heisenberg subalgebras of $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{a}_3$ (up to equivalence with respect to conjugation).*

The homogeneous subalgebras have already been described. For $\mathfrak{a}_1$, there is only one nontrivial element of the Weyl group, namely, r_α. It is easily verified that

$$\tilde{w} = \mathrm{Ad}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma = \sigma_{(1,1)} = \psi\tilde{w}\psi^{-1} = \mathrm{Ad}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

where

$$\psi = \mathrm{Ad}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is the conjugating automorphism.[2] Furthermore,

$$\psi(h) = \psi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = x + y.$$

Now,

$$d_{(1,1)}x = N\gamma_{(1,1)} \cdot \pi_\Delta(x) = 2\left\langle \tfrac{1}{4}\alpha, \alpha \right\rangle x = x.$$

Similarly,

$$d_{(1,1)}y = -y.$$

[2]The choice of matrix representation for ψ is not unique and, in order to keep the entries of the various matrices reasonably straightforward, no attempt has been made to restrict ψ to lie in Ad $SL(n+1)$.

Thus, x has degree 1 and y degree -1. On lifting to $\mathfrak{g}^{(1)}$, the Heisenberg subalgebra is seen to be generated by the elements

$$\lambda^n \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$$

which have degree $2n+1$, $n \in \mathbb{Z}$. This is the well known principal Heisenberg subalgebra of $\mathfrak{a}_1^{(1)}$. In general, the adjective *principal* is applied to the Heisenberg subalgebra associated with the Coxeter class of $\mathfrak{g}$.

For $\mathfrak{a}_2$, the first conjugacy class is that of $w = r_{\alpha_1}$, which was shown to have $\mathsf{s}[w] = (3,2,-1)$. The conjugating automorphism ψ may either be found directly or by embedding the conjugating automorphism for r_α in $\mathfrak{a}_1$. Either way, this leads to

$$\psi = \mathrm{Ad} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that $\sigma = \psi \tilde{w} \psi^{-1}$, where $\sigma = \mathrm{Ad}\ \mathrm{diag}\ (i, -i, 1)$. To find the Heisenberg subalgebra, we examine $\psi(\mathfrak{h})$ using the eigenvectors of w as a basis for $\mathfrak{h}$. Corresponding to the eigenvalue 1 is $h_1 + 2h_2$ and to -1 is h_1. Appealing to the fact that under the standard $\mathfrak{sl}(n+1)$ representation of $\mathfrak{a}_n$ an element is regular if and only if it has distinct eigenvalues [28], so that the diagonal matrices, the elements of $\mathfrak{h}$, are regular if and only if they have distinct entries, it follows that $h_1 + 2h_2 = \mathrm{diag}\ (1,1,-2)$ is not regular while $h_1 = \mathrm{diag}\ (1,-1,0)$ is. Moreover:

$$\psi(h_1 + 2h_2) = h_1 + 2h_2, \quad \psi(h_1) = x_1 + y_1.$$

For $\mathsf{s}[w] = (3,2,-1)$, x_1 has degree 2 and y_1 degree -2, so that the Heisenberg subalgebra is generated by the elements

$$\lambda^n (h_1 + 2h_2) \text{ with degree } 4n,$$

and

$$\lambda^n (\lambda y_1 + x_1) \text{ with degree } 4n+2,$$

the latter of which are the regular elements.[3]

[3]Regular elements of $\mathfrak{g}$ lift into regular elements of $\mathfrak{g}^{(1)}$. See [14, §2] for further discussion.

For the Coxeter class, $w_C = r_{\alpha_1} r_{\alpha_2}$, the automorphism of Theorem 3.3 has corresponding $\mathbf{s}[w_C] = (1,1,1)$, and $\sigma = \text{Ad diag}\,(\omega^2, \omega, 1) = \psi\tilde{w}\psi^{-1}$, where $\omega = \exp(\frac{2\pi i}{3})$ and

$$\psi = \text{Ad} \begin{pmatrix} 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Again, the eigenvectors of w may be used as a basis for $\mathfrak{h}$, namely, $h_1 - \omega h_2$ for the eigenvalue ω and $h_1 - \omega^2 h_2$ for the eigenvalue ω^2, both of which are regular, so that:

$$\psi(h_1 - \omega h_2) = x_0 + x_1 + x_2, \quad \psi(h_1 - \omega^2 h_2) = y_0 + y_1 + y_2.$$

Lifting to $\mathfrak{g}^{(1)}$ with the gradation given by $\mathbf{s}[w_C]$, the principal Heisenberg subalgebra is seen to consist of the elements

$$\lambda^n(\lambda x_0 + x_1 + x_2) \text{ with degree } 3n+1,$$

and

$$\lambda^n(y_0 + \lambda y_1 + \lambda y_2) \text{ with degree } 3n+2,$$

both of which are regular by virtue of the regularity of the respective preimages under ψ.

For $\mathfrak{a}_3$, it has already been established that $w = r_{\alpha_1}$ admits no regular eigenvectors. The Heisenberg subalgebra $\mathcal{H}[r_{\alpha_1}]$ is found in similar fashion to above. In this case, the relevant $\mathbf{s}[w]$-vector is $(3,2,-1,0)$ and $\sigma = \text{Ad diag}\,(i,-i,1,1) = \psi\tilde{w}\psi^{-1}$, where

$$\psi = \text{Ad} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The eigenvectors of w are h_3 and $h_1 + 2h_2$, no linear combination of which is regular, for the eigenvalue 1, and h_1, which is not regular either, for the eigenvalue -1, so that:

$$\psi(h_3) = h_3, \quad \psi(h_1 + 2h_2) = h_1 + 2h_2, \quad \psi(h_1) = x_1 + y_1.$$

It follows that $\mathcal{H}[r_{\alpha_1}]$ is generated by

$$\lambda^n h_3 \text{ and } \lambda^n(h_1 + 2h_2) \text{ with degree } 4n,$$

and

$$\lambda^n(\lambda y_1 + x_1) \text{ with degree } 4n+2,$$

none of which is regular.

Next, for the conjugacy class of $w = r_{\alpha_1} r_{\alpha_3}$, it has been established that $\mathrm{s}[w] = (1,1,-1,1)$ and a simple calculation shows that $\sigma = \mathrm{Ad}\ \mathrm{diag}\ (1,-1,1,-1) = \psi\tilde{w}\psi^{-1}$ where

$$\psi = \mathrm{Ad} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The eigenvectors of w are $h_1 + 2h_2 + h_3$, which is not regular, for the eigenvalue 1, and h_1 and h_3 for the eigenvalue -1. The most general regular eigenvector for the eigenvalue -1 is therefore $h_1 + ch_3$, where $c \neq 0, \pm 1$. Now:

$$\psi(h_1 + 2h_2 + h_3) = h_1 + 2h_2 + h_3, \quad \psi(h_1) = x_1 + y_1, \quad \psi(h_3) = x_3 + y_3.$$

Thus, $\mathcal{H}[w]$ is generated by

$$\lambda^n(h_1 + 2h_2 + h_3) \text{ with degree } 2n,$$

and

$$\lambda^n(\lambda y_1 + x_1) \text{ and } \lambda^n(\lambda y_3 + x_3) \text{ with degree } 2n+1.$$

The regular elements are those of the form

$$\lambda^n(\lambda y_1 + x_1 + c(\lambda y_3 + x_3)), \quad c \neq 0, \pm 1.$$

In the case of $w = r_{\alpha_1} r_{\alpha_2}$, $\mathrm{s}[w] = (2,1,1,-1)$ and $\sigma = \mathrm{Ad}\ \mathrm{diag}\ (\omega, 1, \omega^2, 1) = \psi \tilde{w} \psi^{-1}$, where

$$\psi = \mathrm{Ad} \begin{pmatrix} 1 & \omega & \omega^2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & \omega^2 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The eigenvectors of w are $h_1 + 2h_2 + 3h_3$, which is not regular, for the eigenvalue 1, and $h_1 - \omega h_2$ for the eigenvalue ω and $h_1 - \omega^2 h_2$ for the eigenvalue ω^2, both of which are regular. Furthermore:

$$\begin{aligned} \psi(h_1 + 2h_2 + 3h_3) &= h_1 + 2h_2 + 3h_3, \\ \psi(h_1 - \omega h_2) &= x_1 + x_2 + [y_2, y_1], \\ \psi(h_1 - \omega^2 h_2) &= y_1 + y_2 + [x_1, x_2]. \end{aligned}$$

Thus, $\mathcal{H}[w]$ is generated by:

$$\begin{array}{rll} \lambda^n(h_1 + 2h_2 + 3h_3) & \text{with degree} & 3n, \\ \lambda^n(\lambda[y_2, y_1] + x_1 + x_2) & \text{with degree} & 3n+1, \\ \lambda^n(\lambda y_1 + \lambda y_2 + [x_1, x_2]) & \text{with degree} & 3n+2. \end{array}$$

The last two of these give the regular elements.

Finally, for the Coxeter class with $w_C = r_{\alpha_1} r_{\alpha_2} r_{\alpha_3}$, $\mathrm{s}[w_C] = (1,1,1,1)$ and $\sigma = \mathrm{Ad}\ \mathrm{diag}\ (-i, -1, i, 1) = \psi \tilde{w} \psi^{-1}$, where

$$\psi = \mathrm{Ad}\ \mathrm{diag} \begin{pmatrix} 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The eigenvectors of w are $h_1 + (1-i)h_2 - ih_3$, which is regular, for the eigenvalue i, $h_1 + h_3$, which is not regular, for the eigenvalue -1, and $h_1 + (1+i)h_2 + ih_3$, which is regular, for the eigenvalue $-i$. In this case, the action of ψ on $\mathfrak{h}$ is as follows:

$$\begin{aligned}
\psi(h_1 + (1-i)h_2 - ih_3) &= x_0 + x_1 + x_2 + x_3, \\
\psi(h_1 + h_3) &= [x_1, x_2] + [x_2, x_3] + [y_2, y_1] + [y_3, y_2], \\
\psi(h_1 + (1+i)h_2 + ih_3) &= y_0 + y_1 + y_2 + y_3.
\end{aligned}$$

Thus, the principal Heisenberg subalgebra is generated by:

$$\begin{aligned}
\lambda^n(\lambda x_0 + x_1 + x_2 + x_3) &\quad \text{with degree} \quad 4n+1, \\
\lambda^n([x_1, x_2] + [x_2, x_3] + \lambda[y_2, y_1] + \lambda[y_3, y_2]) &\quad \text{with degree} \quad 4n+2, \\
\lambda^n(y_0 + \lambda y_1 + \lambda y_2 + \lambda y_3) &\quad \text{with degree} \quad 4n+3.
\end{aligned}$$

The regular elements are those of odd degree.

3.4.1 The Bos Algorithm Revisited

In actual fact, the $\mathbf{s}_+$-vector given by the Bos algorithm is not guaranteed to produce a gradation for which the generating elements of $\psi(\mathfrak{h})$ are homogeneous mod N. That this did happen in each of the examples presented was only because the representatives of the conjugacy classes were chosen in order to facilitate a straightforward exposition. In general, given any representative of a conjugacy class, the Bos algorithm yields the same $\mathbf{s}_+$-vector. Provided this $\mathbf{s}_+$-vector is considered as being with respect to an appropriate choice of simple roots, the generating elements of $\psi(\mathfrak{h})$ will all be homogeneous mod N. This basis dependency is implicit in the way the shift vectors were embedded and the Bos algorithm subsequently applied. In the examples, everything was done over the ordered basis of simple roots given by $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

To illustrate the basis dependency, consider the conjugacy class of $r_{\alpha_0} r_{\alpha_1}$ in $\mathfrak{a}_3$, which was shown to have $\mathbf{s}_+ = (1,1,0,1)$. Another element of the class is $w = r_{\alpha_1} r_{\alpha_2}$. Proceeding as in Example 3.4.2, that is with respect to the simple basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, the same $\mathbf{s}_+$-vector results, namely $(1,1,0,1)$. However, the generating elements of $\psi(\mathfrak{h})$ are

$$h_1 + 2h_2 + 3h_3, \quad x_1 + x_2 + [y_2, y_1], \quad y_1 + y_2 + [x_1, x_2],$$

which are not homogeneous mod N ($N = 3$) with respect to $\mathbf{s}_+ = (1,1,0,1)$. Nevertheless, they are homogeneous with respect to $\mathbf{s} = (1,1,1,0)$, which is obtained from $\mathbf{s}_+$ by a symmetry of the Dynkin diagram of $\mathfrak{a}_3^{(1)}$, so that the two corresponding Kač automorphisms are conjugate. On the other hand, using $\{\alpha_0, \alpha_2, \alpha_3\}$ as the ordered basis of simple roots, it follows that

$$\gamma = \tfrac{1}{3}(\alpha_1 + \alpha_2) = -\tfrac{1}{3}(\alpha_0 + \alpha_3),$$

so that $\mathbf{c} = (-\frac{1}{3}, 0, -\frac{1}{3})$ with respect to this basis. Hence

$$(s_0, s_2, s_3) = 3\mathbf{c}A = (-1, 1, -1),$$

where

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

so that $s_1 = 4$, and

$$\mathbf{s} = (s_0, s_1, s_2, s_3) = (-1, 4, 1, -1) \xrightarrow{\text{Bos}} \cdots \longrightarrow \mathbf{s}_+ = (1,1,1,0).$$

Of course, $\mathbf{s}_+ = (s_1, s_2, s_3, s_0) = (1,1,0,1)$ with respect to the ordered basis of simple roots $\{\alpha_2, \alpha_3, \alpha_0\}$.

In other words, the $\mathbf{s}_+$-vector given by the Bos algorithm will always work, provided we interpret it over an appropriate choice of simple roots. As another example, consider the class of r_{α_0} for $\mathfrak{a}_3$. Another representative is $w = r_{\alpha_1+\alpha_2}$. Consider the ordered simple root basis given by

$$\beta_1 = -\alpha_1 - \alpha_2 - \alpha_3, \quad \beta_2 = \alpha_2 + \alpha_3, \quad \beta_3 = -\alpha_2,$$

for which $\alpha_1 + \alpha_2 = -\beta_1 - \beta_2 - \beta_3 = \beta_0$. Embedding the shift vector and applying the Bos algorithm in exactly the same way as in the Example 3.4.2 for r_{α_0}, it follows that $s_+ = (2,1,0,1)$ with respect to $\{\beta_1, \beta_2, \beta_3\}$. The generators of $\psi(\mathfrak{h})$, which are

$$h_1 - h_2, \quad h_1 + h_2 + 2h_3, \quad [x_1, x_2] + [y_2, y_1],$$

are now homogeneous mod N ($N = 4$) (their repective degrees being 0, 0 and 2 mod 4) with respect to the $s_+ = (2,1,0,1)$ gradation over the β-basis. Expressed in terms of the original α-basis, this vector has negative entries, but may be transformed to $\mathbf{s} = (2,1,0,1)$ (with respect to the α-basis) by the Bos algorithm. The sequence of steps required reflects the change of basis necessary to pass from the β-basis to the α-basis.

3.5 Comments

In the examples presented, ψ has been chosen so that $\psi(\mathfrak{h})$, which lifts to the Heisenberg subalgebra $\mathcal{H}[w]$ of $\mathfrak{a}_n^{(1)}$, consists of the generators of the principal Heisenberg subalgebras of the factors $\mathfrak{a}_{n_k}$ along with $\mathfrak{h}'$. For such choice of ψ, $\psi(\mathfrak{h})$ is contained in the subalgebra of fixed points of $\tilde{w}$. Moreover, $\psi_{|\mathfrak{a}_{n_k}} : \mathfrak{a}_{n_k} \to \mathfrak{a}_{n_k}$ so that ψ preserves the orthogonal decomposition of the regular subalgebra of $\mathfrak{a}_n$ associated with w. That such a decomposition may always be found is shown as follows.

First, note that the principal Heisenberg subalgebra of a simple Lie algebra $\mathfrak{g}$ may be generated by the fixed points of $\tilde{w}_C$, the extension of the Coxeter element to Aut($\mathfrak{g}$). This is due to a result of Kostant [38]: the fixed points of $\tilde{w}_C$ span a CSA of $\mathfrak{g}$. (Of course, there are subalgebras isomorphic to the principal Heisenberg subalgebra, and these are generated by the fixed points of automorphisms conjugate to $\tilde{w}_C$.) Now assume $w \in W(\mathfrak{a}_n)$ is of non-Coxeter type. From the product decomposition $w = w_1 \dots w_r$, where w_k is the Coxeter element of the factor $\mathfrak{a}_{n_k}$ in the corresponding regular subalgebra of $\mathfrak{a}_n$, it follows that the span $\bar{\mathfrak{h}}_{n_k}$ of generators of the principal Heisenberg subalgebra of $\mathfrak{a}_{n_k}$ consists entirely of fixed points of $\tilde{w}$.

Theorem 3.5.1 *The Heisenberg subalgebra $\mathcal{H}[w]$ is generated by*

$$\bar{\mathfrak{h}} := \bigoplus_k \bar{\mathfrak{h}}_{n_k} \oplus \mathfrak{h}'$$

which is contained in the subalgebra of fixed points of $\tilde{w}$.

PROOF: The latter statement is obvious in light of the preceding remarks and the fact that $\tilde{w}$ acts as the identity on $\mathfrak{h}'$. To show that $\bar{\mathfrak{h}}$ generates $\mathcal{H}[w]$, it is enough to establish that $\bar{\mathfrak{h}}$ is a CSA of $\mathfrak{g}$, given the original definition of Heisenberg subalgebras as sections of bundles over the variety of CSA's of a Lie algebra [37]. This means we need to show that $\bar{\mathfrak{h}}$ is a nilpotent subalgebra that is self-normalizing.[4] Clearly, $\bar{\mathfrak{h}}$ is Abelian and therefore nilpotent, since each $\bar{\mathfrak{h}}_{n_k}$ is Abelian, being the generating set of the principal Heisenberg subalgebra of $\mathfrak{a}_{n_k}$; furthermore, because of the orthogonal decomposition of the regular subalgebra, $\mathfrak{h}'$ is orthogonal to each $\mathfrak{h}_{n_k}$ and hence to each $\mathfrak{a}_{n_k}$.

Now consider the normalizer in $\mathfrak{a}_n$ of $\bar{\mathfrak{h}}$:

$$N_{\mathfrak{a}_n}(\bar{\mathfrak{h}}) := \left\{x \in \mathfrak{a}_n : [x, \bar{\mathfrak{h}}] \subset \bar{\mathfrak{h}}\right\}$$

By construction, each element of $\bar{\mathfrak{h}}$ is a fixed point of $\tilde{w}$. Now, $\mathfrak{a}_n$ is spanned by the eigenvectors of $\tilde{w}$, so it suffices to consider x as an eigenvector of $\tilde{w}$ corresponding to some eigenvalue ϵ^j, where $\epsilon = \exp(\frac{2\pi i}{N})$, N being the order of $\tilde{w}$.

Note that the elements of $\bigoplus_k \bar{\mathfrak{h}}_{n_k}$ are of the form $\sum_\beta E_\beta$ where $E_\beta \in L^\beta$, as we have chosen our lifts $\tilde{w}$ to map $E_\beta \mapsto cE_{w(\beta)}$ where $c = 1$. Moreover, $\sum \beta = 0$ as each $\bar{\mathfrak{h}}_{n_k}$ gives all the fixed points of the extension of the Coxeter element for $\mathfrak{a}_{n_k}$, of which there are none in $\mathfrak{h}_{n_k}$ itself, Coxeter elements being primitive. This means that any fixed points of Coxeter extensions must lie in the lift of the zero root (the trivially fixed point of any linear map) in the sense above that $\sum \beta = 0$.

First, consider those $x \in \mathfrak{a}_n \setminus \bar{\mathfrak{h}}$ that are fixed points of $\tilde{w}$ (such elements exist since w is non-Coxeter), and write

$$x = \sum_\beta E_\beta, \quad E_\beta \in L^\beta.$$

[4]Usually, the definition of a CSA for a semisimple Lie algebra is a maximal subalgebra, each of whose elements is ad-semisimple [33]. Here we are appealing to the more general definition for arbitrary Lie algebras that defines a CSA as being a nilpotent self-normalizing subalgebra. This encompasses the previous requirements in the semisimple case [33].

Furthermore, a basis for such x is given by those x for which

$$\sum \beta = (n+1)\lambda_k \in \mathfrak{h}'^*,$$

for appropriate fundamental dominant weight λ_k (namely, k such that r_{α_k} does not appear in w), as $\mathfrak{h}'^*$ consists of the fixed points of w whose extension preserves the action on $\mathfrak{h}^*$. This means β is a positive integral sum of simple roots. Next, take $h \in \mathfrak{h}'$ corresponding to $(n+1)\lambda_k \in \mathfrak{h}'^*$. Then

$$[x, h] = \sum_\beta [E_\beta, h] = \sum_\beta \langle \lambda_k, \beta \rangle E_\beta,$$

where the last sum need only be taken over those roots β consisting of strings containing the simple root α_k as $\langle \lambda_k, \alpha_j \rangle = \delta_{jk}$. As each of the terms $\langle \lambda_k, \beta \rangle \neq 0$ and the sum of such β is nonzero, it follows that $[x, h]$ must be a *nonzero* fixed point of $\tilde{w}$ *not* in $\bar{\mathfrak{h}}$. Thus, $x \notin N_{\mathfrak{a}_n}(\bar{\mathfrak{h}})$.

Second, consider the case when x is an eigenvector with eigenvalue not equal to 1. There are two possibilities: Either $x = \sum_\beta c_\beta E_\beta$ where the c_β are Nth roots of unity (not all 1) and $\sum \beta = (n+1)\lambda_k$, for some k, whence a similar argument to above shows that $0 \neq [x, h] \notin \bar{\mathfrak{h}}$, where h corresponds to $(n+1)\lambda_k \in \mathfrak{h}'$. Alternatively, $x \in \mathfrak{a}_{n_k}$, whence $[x, y]$ lies in the same eigenspace as x and is nonzero for suitable $y \in \bar{\mathfrak{h}}_{n_k}$ by maximality of the known CSA $\bar{\mathfrak{h}}_{n_k}$ of $\mathfrak{a}_{n_k}$. This shows that $x \notin N_{\mathfrak{a}_n}(\bar{\mathfrak{h}})$ when x is an eigenvector that is not a fixed point, thus completing the proof. □

The method presented here for constructing Heisenberg subalgebras may be summarised thus:

- Identify the conjugacy classes of the Weyl group.
- Select a representative of each class and lift it to $\mathrm{Aut}(\mathfrak{g})$.
- Embed the corresponding shift vector by identifying the regular subalgebra.
- Take each of the principal Heisenberg subalgebras of the components of the regular subalgebra $\mathfrak{a}_{n_1} \oplus \cdots \oplus \mathfrak{a}_{n_r}$ along with $\mathfrak{h}' \otimes \mathbb{C}[\lambda, \lambda^{-1}]$.
- This gives a Heisenberg subalgebra whose elements are homogeneous with respect to the gradation induced by the above shift vector.

- There exists a conjugating map, ψ, which preserves the orthogonal decomposition induced by w and for which $\psi(\mathfrak{h})$ generates the Heisenberg subalgebra and consists of fixed points of $\tilde{w}$.
- Use the Bos algorithm to find the conjugate Kač automorphism and associated vector, $\mathbf{s}_{+}$, from which the precise structure of the fixed point subalgebra of $\tilde{w}$ may be determined. Note that in general, the Heisenberg subalgebra will not be homogeneous with respect to the $\mathbf{s}_{+}$-gradation.

Chapter 4

Generalized Drinfel'd-Sokolov Hierarchies

In this chapter, the theory of generalized Drinfel'd-Sokolov (GD-S) hierarchies is recapitulated. The necessary essentials from [13] are outlined in §4.1. In §4.2, it is proved that in the case of $\mathfrak{a}_n$, Weyl group elements admitting a regular eigenvector yield a regular Heisenberg element of degree 1 if the order is preserved on lifting, and a regular Heisenberg element of degree 2 if the order doubles. Finally, in §4.3, the theory of gauge symmetries necessary for specializations, as originally formulated by Guil [25], is generalized for arbitrary Heisenberg subalgebras admitting a regular eigenvector.

4.1 Background

In their seminal papers, [18, 17], Drinfel'd and Sokolov sucessfully incorporated the then existing knowledge of Zakharov's and Shabat's Lax pairs [59] and many seemingly disparate areas relating to so-called integrable systems, and put them firmly within the context of zero-curvature equations over affine Lie algebras, which were developed by Kač and Moody [35, 36], the necessary fundamentals of which have been described in the preceding chapters. Among the many interesting features of integrable systems, appear the existence of soliton solutions, conservation laws, prolongation structures, Bäcklund transformations and Miura maps, the latter taking their nomenclature from the most celebrated of equations in this rich field, the Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) equations which were long known to be intimately connected to $\mathfrak{a}_1$. Drinfel'd and Sokolov were able to construct systems analogous to the KdV and mKdV equations for *arbitrary* untwisted and twisted loop algebras over the simple Lie algebras. See also [56]. Those of KdV-type are connected with the homogeneous Heisenberg subalgebra, those of mKdV-type with the principal Heisenberg subalgebra, respectively. In [13], de Groot *et al.*, inspired by the earlier work of Wilson [55], were able to generalize even further the ideas of Drinfel'd and Sokolov, by associating zero-curvature systems to the remaining inequivalent Heisenberg subalgebras, which as has been seen, are in one-to-one correspondence with the conjugacy classes of $W(\mathfrak{g})$. They were able also to generalize the well known concept of the Miura transformation linking the mKdV and KdV equations by introducing a partial order on the gradations of $\mathfrak{g}^{(1)}$. Subsequent papers have taken these ideas even further [6, 7, 30, 31, 4, 5, 20].

First then, we define the partial ordering of gradations of $\mathfrak{g}^{(1)}$. This refers to *all* possible gradations $\mathbf{s}$, not just those of the form $\mathbf{s}[w]$ associated to a conjugacy class of w.

Definition 4.1.1 *For any two gradations of* $\mathbf{s}_1, \mathbf{s}_2$ *of* $\mathfrak{g}^{(1)}$, *we say that* $\mathbf{s}_1 \prec \mathbf{s}_2$ *if*

$$\begin{aligned} \mathfrak{g}_0^{(1)}(\mathbf{s}_2) &\subset \mathfrak{g}_0^{(1)}(\mathbf{s}_1), \\ \mathfrak{g}_{>0}^{(1)}(\mathbf{s}_2) &\supset \mathfrak{g}_{>0}^{(1)}(\mathbf{s}_1), \\ \mathfrak{g}_{<0}^{(1)}(\mathbf{s}_2) &\supset \mathfrak{g}_{<0}^{(1)}(\mathbf{s}_1). \end{aligned}$$

Clearly, $\prec$ is a partial order on the set of gradations of $\mathfrak{g}^{(1)}$. The relationship $\mathbf{s}_1 \prec \mathbf{s}_2$ may be represented diagrammatically as in Figure 4.1.

The gradations given in Tables 2.1 and 2.2 are partially ordered with respect to increasing subscripts.

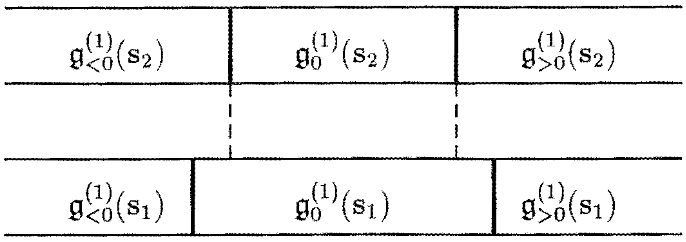

Figure 4.1: $\mathrm{s}_1 \prec \mathrm{s}_2$

Now we may define the generalized Drinfel'd-Sokolov heirarchies. First we set up some Lax operators

$$\mathcal{L}_q : C^\infty(\mathbb{R}, \mathfrak{g}^{(1)}) \to C^\infty(\mathbb{R}, \mathfrak{g}^{(1)}).$$

Definition 4.1.2 *A Heisenberg element $\Lambda \in \mathcal{H}[w]$ is said to be* regular *if $\mathcal{H}[w] =$ Ker ad Λ.*

Remark: If Λ is *not* regular, then Λ lies in a hyperplane orthogonal to some root α, where $[\Lambda, L^\alpha] = 0$ and $\alpha \in$ Ker ad Λ so that $\mathcal{H}[w]$ is properly contained in the subspace Ker ad Λ. □

Definition 4.1.3 *Given $w \in W(\mathfrak{g})$, with associated gradation $\mathrm{s}[w]$ of $\mathfrak{g}^{(1)}$, let Λ be a Heisenberg element of fixed degree $j > 0$. For any gradation s of $\mathfrak{g}^{(1)}$ such that $\mathrm{s} \prec \mathrm{s}[w]$, define $q \in C^\infty(\mathbb{R}, Q)$ where*

$$Q := \mathfrak{g}^{(1)}_{\geq 0}(\mathrm{s}) \cap \mathfrak{g}^{(1)}_{<j}(\mathrm{s}[w]).$$

Then

$$\mathcal{L}_q := D_x + q + \Lambda$$

is said to be of Type I if Λ is regular and of Type II otherwise.

In most of the existing literature, it is the hierarchies of Type I that are considered. Those of Type II have been discussed in [22, 1, 2].

REMARK: Strictly speaking, in order for the Lax operator to be properly defined, we identify Λ with the constant valued map $\mathbb{R} \to \{\Lambda\}$. $\square$

When $\mathbf{s} = \mathbf{s}[w]$, $\mathcal{L}_q$ is called the generalized modified KdV Lax operator. When $\mathbf{s}$ is minimal, that is $s_k = 1$ for some k and $s_j = 0$ for all $j \neq k$, $\mathcal{L}_q$ is called a generalized KdV Lax operator. Note that for $\mathfrak{g}^{(1)} = \mathfrak{a}_n^{(1)}$, there is only one minimal gradation, up to equivalence, namely the homogeneous gradation $\mathbf{s} = (1, 0, \ldots, 0)$. This is due to the cyclic symmetry of the extended Dynkin diagram of $\mathfrak{a}_n^{(1)}$. Finally, if $\mathbf{s}$ is any other "intermediate" gradation, that is not minimal and $\mathbf{s} \neq \mathbf{s}[w]$, then $\mathcal{L}_q$ is called the generalized *partially* modified KdV Lax operator.

When $\mathbf{s} = \mathbf{s}[w]$, the generalized modified KdV hierarchy results from the zero-curvature equation

$$[\mathcal{L}_q, D_t + a(q) + \Lambda'] = 0$$

where

$$a(q) = \sum_{j=0}^{k-1} a_j(q),$$

where each $a_j(q) \in C^\infty(\mathbb{R}, \mathfrak{g}_j^{(1)}(\mathbf{s}[w]))$ and $\Lambda' \in \mathcal{H}_k[w]$, $k > 0$. Without loss of generality, the component of q of $\mathbf{s}[w]$-degree $j-1$ lies in $\mathrm{Im\ ad}\ \mathcal{H}[w]$.

More generally, for $\mathbf{s} \prec \mathbf{s}[w]$, the partial hierarchies result from a "gauge fixed" zero-curvature condition

$$[\mathcal{L}_j, \mathcal{L}_k] = 0$$

where $\mathcal{L}_j = D_x + \tilde{q} + \Lambda$, $\Lambda \in \mathcal{H}_j[w]$ and $\tilde{q}$ is a certain "gauge fixed" element in

$$C^\infty\left(\mathbb{R}, \mathfrak{g}_{\geq 0}^{(1)}(\mathbf{s}) \cap \mathfrak{g}_{<j}^{(1)}(\mathbf{s}[w])\right).$$

Similar gauge fixing conditions are imposed on $\mathcal{L}_k$. These arise due to a freedom resulting in the subspace

$$P := \mathfrak{g}_0^{(1)}(\mathbf{s}) \cap \mathfrak{g}_{<0}^{(1)}(\mathbf{s}[w]).$$

Such freedom does not arise when $\mathbf{s} = \mathbf{s}[w]$ as then $P = 0$.

In [13], de Groot *et al.* gauge transform the Lax operator $\mathcal{L}_q$ of Type I to

$$\mathcal{L}_0 = \exp(\mathrm{ad}\ T)(\mathcal{L}_q) = D_x + \sum_{l<j} H_l + \Lambda, \tag{4.1}$$

where $H_l \in C^\infty(\mathbb{R}, \mathcal{H}_l[w])$ for $l < j$ and $T \in C^\infty(\mathbb{R}, \mathfrak{g}^{(1)}_{<0}(\mathrm{s}[w])$. This may be regarded as analogous to a diagonalization procedure insofar as

$$\mathcal{L}_0 = (\exp T)\mathcal{L}_q(\exp T)^{-1}.$$

When w is the identity element, so that $\mathcal{H}[w]$ is the homogeneous Heisenberg subalgebra, that is, the loop algebra over the CSA $\mathfrak{h}$, then each H_l appears in the standard matrix representation as a *diagonal* matrix.

In Proposition 3.5 of [13], it is shown that the H_l are conserved densities for the hierarchy, by which is meant for each l, there exists $A_l \in C^\infty(\mathbb{R}, \mathcal{H}_l[w])$ such that

$$D_t H_l + D_x A_l = 0, \ \ l < j.$$

Furthermore, in Proposition 3.6 of [13], it is shown that $D_x A_l = 0$ for $0 \leq l < j$, so that the corresponding H_l are constant along the flows, $D_t H_l = 0$ for $0 \leq l < j$. Therefore, without loss of generality, the H_l may be set to zero for $0 \leq l < j$. In particular, by extracting the $(j-1)$-degree $\mathrm{s}[w]$-component of (4.1),

$$H_{j-1} + [\Lambda, T_{-1}] = q_{j-1},$$

where T_{-1} is the (-1)-degree $\mathrm{s}[w]$-component of T. Since H_{j-1} may be set to zero, it follows that:

Proposition 4.1.4 *If $\mathcal{L}_q$ is a Lax operator of Type I, so that*

$$\mathcal{L}_q = D_x + q + \Lambda,$$

where Λ is a regular element of the Heisenberg subalgebra $\mathcal{H}[w]$, then, without loss of generality, q_{j-1}, the $(j-1)$-degree $\mathrm{s}[w]$-component of q, may be constrained to lie in $C^\infty(\mathbb{R}, Im\ ad\ \mathcal{H}[w])$. In other words, q_{j-1} may be assumed to have no $\mathcal{H}[w]$ component.

We shall restrict ourselves to the generalized modified hierarchies when developing our specialization theory.

Finally, for the two hierarchies related to $\mathfrak{s}_2$ and $\mathfrak{s}_1 \prec \mathfrak{s}_2 \prec \mathfrak{s}[w]$, de Groot *et al.* show that there is a transformation taking solutions of the $\mathfrak{s}_2$ hierarchy to those of the $\mathfrak{s}_1$ hierarchy. This generalizes the Miura transformation.

REMARK: The work of de Groot *et al.* [13] is not the only approach to generalizing Drinfel'd-Sokolov hierarchies. Most notably, there is that of [43], using the theory of algebraic curves and flag manifolds and drawing on the work of Segal, Wilson and others [49, 57, 58, 48]. □

4.2 $\mathcal{H}[w]$ Elements of Minimal Positive Degree

In order to obtain specializations of the generalized Drinfel'd-Sokolov hierarchies, it is necessary to make a distinction as to whether or not the order of w doubles on lifting to $\tilde{w}$. We shall only concern ourselves with those Λ of minimal positive degree for the $\mathfrak{s}[w]$-gradation.

Note that if $\mathcal{H}_1[w] \neq \emptyset$ so that $1 \in I[w]$, then $\exp(\frac{2\pi i}{N})$ is an eigenvalue of w by definition of $I[w]$ (N being the order of $\tilde{w}$). Thus,

$$\exp\left(\tfrac{2\pi i}{N}\right) = \exp\left(\tfrac{2\pi i m}{n}\right)$$

for some $m \in \mathbb{Z}_n$, where n is the order of w itself. As N is either n or $2n$, it follows that the only possibility is $N = n$ and $m = 1$. In other words, if $\mathcal{H}_1[w] \neq \emptyset$, then the order of w is preserved on lifting.

For $\mathfrak{g} = \mathfrak{a}_s$, the converse is true for those w admitting a regular eigenvector. This may be seen by recalling that such w arise from the partitions $\mathcal{P}_1 = \{p, \ldots, p\}$ or $\mathcal{P}_2 = \{p, \ldots, p, 1\}$ (both sets with r elements) of the integer $s+1$. For $\mathcal{P}_1$, the regular subalgebra is

$$\mathfrak{a}_{p-1} \oplus \cdots \oplus \mathfrak{a}_{p-1}$$

where $s = pr - 1$, with CSA decomposition

$$\mathfrak{h} = \mathfrak{h}_{p-1} \oplus \cdots \oplus \mathfrak{h}_{p-1} \oplus \mathfrak{h}'_{r-1},$$

and $w = w_1 \cdots w_r$, where w_l is the Coxeter element for $\mathfrak{a}_{p-1}$ for all l and thus has order p. Hence, w itself has order $n = p$ and has eigenvalues given by those of each of the w_l as well as 1 if $\mathfrak{h}_{r-1} \neq \{0\}$, *i.e.* $r \neq 1$. Thus the eigenvalues of w are just $\omega^1, \ldots, \omega^{p-1}$ and maybe 1, where $\omega = \exp(\frac{2\pi i}{p})$, since these are precisely the eigenvalues of the Coxeter element of $\mathfrak{a}_{p-1}$ [55]. In particular, $\omega^1 = \exp(\frac{2\pi i}{n})$ is an eigenvalue of w. A similar argument shows this to be true for $\mathcal{P}_2$ as well. Therefore, if the order is preserved on lifting, so that $N = n$, it follows that $\exp(\frac{2\pi i}{N})$ is an eigenvalue of w so that $1 \in I[w]$ and $\mathcal{H}_1[w] \neq \emptyset$.

Thus in the regular case, $w : \mathfrak{h} \to \mathfrak{h}$ has order p and eigenvalues of all orders ω^l, $l = 1, \ldots, p-1$, as well as a possible eigenvalue of 1. Moreover, if $\tilde{w}$ has double the order of w, so that $N = 2n = 2p$, then it has eigenvectors in $\mathfrak{h}$ with eigenvalues

$$\omega^l = \exp\left(\tfrac{2\pi i}{n}\right)^l = \exp\left(\tfrac{2\pi i}{N}\right)^{2l}, \quad l = 1, \ldots, p-1.$$

Applying the conjugating map ψ to obtain the corresponding homogeneous elements of $\mathcal{H}[w]$, it follows that $\mathcal{H}[w]$ consists of elements of all *even* degree, with the possible exception of elements of degree zero. However, the only case where $0 \notin I[w]$ is for $\mathcal{P}_1$ when $r = 1$ so that $\mathfrak{a}_s = \mathfrak{a}_{p-1}$ and $w = w_C$, which does not double in order. To summarise then, we have established:

Proposition 4.2.1 *Let $\mathfrak{g}$ be a Lie algebra of type $\mathfrak{a}$ and suppose $w \in W(\mathfrak{g})$ admits a regular eigenvector. Then:*
(i) the order of w is preserved on lifting to $\tilde{w} \in Aut(\mathfrak{g})$ if and only if $\mathcal{H}_1[w] \neq \emptyset$;
(ii) if the order of w doubles on lifting, then $\mathcal{H}[w]$ consists of elements of all even degrees.

In fact, it is always guaranteed that when w, of order n, admits a regular eigenvector, a regular eigenvector may be found for the eigenvalue $\omega = \exp(\frac{2\pi i}{n})$, as is discussed in §4 of [15]. Therefore:

Proposition 4.2.2 *Let $\mathfrak{g}$ be a Lie algebra of type $\mathfrak{a}$ and suppose $w \in W(\mathfrak{g})$ admits a regular eigenvector. Then there exists a regular Heisenberg element, Λ, of minimal positive degree so that $\mathcal{H}[w] = Ker\ ad\ \Lambda$.*

4.3 Gauge Symmetries

We are now in a position to extend the work of Guil [25] by setting it within the framework of generalized Drinfel'd-Sokolov hierarchies as expounded by de Groot *et al.* in [13].

In the notation of [13], the ideas of Guil [25] may be expressed as follows, bearing in mind that Guil takes $\mathbf{s}[w] = (1, \ldots, 1)$, thus dealing with the principal Heisenberg subalgebra. Given $w \in W(\mathfrak{g})$, with associated Heisenberg subalgebra $\mathcal{H}[w]$ homogeneous with respect to the gradation induced by $\mathbf{s}[w]$, we define a differential operator as in [13, §3.1],

$$\mathcal{L}_q : C^\infty(\mathbb{R}, \mathfrak{g}^{(1)}) \to C^\infty(\mathbb{R}, \mathfrak{g}^{(1)}),$$

by

$$\mathcal{L}_q := D_x + q + \Lambda,$$

where $q \in C^\infty(\mathbb{R}, Q)$, $Q := \mathfrak{g}^{(1)}_{\geq 0}(\mathbf{s}[w]) \cap \mathfrak{g}^{(1)}_{<j}(\mathbf{s}[w])$ and $\Lambda \in \mathcal{H}_j[w]$ is regular.[1] Here we are taking $\mathbf{s} = \mathbf{s}[w]$ in [13] so that we are restricted to the so-called generalized Drinfel'd-Sokolov modified-KdV hierarchies.

A comparison with [25] reveals that q corresponds to Guil's entity u and Λ to λF. Note that in [25], everything is done in the affine algebra $L(\mathfrak{g}, \sigma)$, where, for $\mathfrak{g} = \mathfrak{a}_n$, $\sigma = \mathrm{Ad}\ \mathrm{diag}(\omega^n, \ldots, \omega, 1)$, with $\omega = \exp(\frac{2\pi i}{n+1})$, is the automorphism of type $(\mathbf{s}[w_C]; 1) = (1, \ldots, 1; 1)$, which is conjugate to the Coxeter automorphism. Hence, $L(\mathfrak{g}, \sigma)$, with homogeneous gradation given by the derivation $d = \lambda \frac{d}{d\lambda}$ (*i.e.* any element in $\lambda^l \mathfrak{g}_{l \bmod N}$ has degree l, where N is the order of w_C) is isomorphic to $\mathfrak{g}^{(1)}$ equipped with the $\mathbf{s}[w_C]$-gradation.

The zero-curvature equations in [25] (2.3) become

$$[\mathcal{L}_q, D_t - A] = 0, \tag{4.2}$$

as in [13] (3.8) where $A = \exp(-\mathrm{ad}\, T)(b)_+$ is the V^+ entity in [25], the +-subscript/

[1] For technical reasons, some authors work in $C^\infty(S^1, \mathfrak{g}^{(1)})$.

superscript denoting the projection onto the subalgebra of $\mathfrak{g}^{(1)}$ of nonnegative $\mathbf{s}[w]$-grade. Here $T = \sum_{l<0} T_l$ where T_l has $\mathbf{s}[w]$-grade l and b is a Heisenberg subalgebra element of degree $k \geq j$, corresponding to v_0 in [25]. Thus,

$$A \in \mathfrak{g}^{(1)}_{\geq 0}(\mathbf{s}[w]) \cap \mathfrak{g}^{(1)}_{\leq k}(\mathbf{s}[w]).$$

As in [25], we introduce the notion of a "differential grading" following the earlier work of Wilson [55]. This assigns a degree of 1 to the components, q_l, of $q = \sum_l q_l Z_l$, where the Z_l comprise a basis of Q formed from the usual root space decomposition with respect to $\mathfrak{h}$, and degree $m > 0$ to $q_l^{(m)} := D_x^m q_l$. This differential grading on the jet bundle over the q_l is then extended to $C^\infty(\mathbb{R}, \mathfrak{g}^{(1)})$ by specifying the constant elements, namely those of $\mathfrak{g}^{(1)}$, to have a differential degree of zero. Note that from the equations arising from (4.2) it follows that the $\mathbf{s}[w]$-grade component of A of degree l must be homogeneous of differential degree $k - l$ [55].

Now, construct the isomorphic jet bundle over variables $\bar{q}_l$ and consider the parallel picture of zero-curvature equations involving the operator $\mathcal{L}_{\bar{q}} = D_x + \bar{q} + \Lambda$, where $\bar{q} = \sum_l \bar{q}_l Z_l$, so that everything is exactly the same in the $\mathcal{L}_{\bar{q}}$ picture as it is for $\mathcal{L}_q$, except for the relabelling of the variables with a "bar" above them. Obviously, the zero-curvature equations obtained in each case are identical apart from this relabelling. Eventually, we will impose a specialization involving a relationship between $\bar{q}$ and q — one will be the image of the other under a certain automorphism of $\mathfrak{g}^{(1)}$. From now on, we shall be working in the algebra $B = \mathbb{C}[q_l^{(m)}, \bar{q}_l^{(m)}] \otimes \mathfrak{g}^{(1)}$ with the differential grading assigning both $q_l^{(m)}$ and $\bar{q}_l^{(m)}$ the differential degree m. It may be necessary on occasion to step up to the algebra $\bar{B}$, containing B, for which the derivation D_x is surjective. In other words, $\bar{B}$ is the nonlocal extension of B containing integrals of its elements. Its existence in this context is discussed in some detail by Wilson [55, 54]. The introduction of $\bar{B}$ is, for our purposes, a necessary formalism. However, any potential nonlocality is beyond the scope of this work. Such a situation might arise in solving certain equations which require the integration of an expression that is not a total derivative in $q_l^{(m)}$, $\bar{q}_l^{(m)}$ for $m \geq 0$. To our knowledge, no attempt has been made to put this on a more rigorous footing, although Wilson [55, 54] has examined the situation for $\mathbb{C}[q_l^{(m)}] \otimes \mathfrak{g}^{(1)}$ for some of the cases under present consideration.

Now we introduce the notion of a "gauge symmetry" of our zero-curvature system of equations. This is discussed in Proposition 3.3 of [13], which determines the most general form of gauge transformation that preserves the form of $\mathcal{L}_q$. It shows that for systems of the type $(\Lambda, w, \mathbf{s})$, where $\mathbf{s} \prec \mathbf{s}[w]$,

$$\mathcal{L}_{\bar{q}} = \exp(\text{ad } S)(\mathcal{L}_q),$$

where $S \in C^\infty(\mathbb{R}, P)$, with

$$P = \mathfrak{g}_0^{(1)}(\mathbf{s}) \cap \mathfrak{g}_{<0}^{(1)}(\mathbf{s}[w]).$$

However, the authors appear to be unnecessarily restrictive in their proof insofar as they require that

$$[S, \mathcal{L}_q] = -D_x S + [S, q + \Lambda] \in C^\infty(\mathbb{R}, Q), \tag{4.3}$$

for

$$\begin{aligned} \bar{q} &= \exp(\text{ad } S)(\mathcal{L}_q) - D_x - \Lambda \\ &= q + [S, \mathcal{L}_q] + \frac{1}{2!}[S, [S, \mathcal{L}_q]] + \cdots. \end{aligned} \tag{4.4}$$

This leads to S taking values in P. For our purposes, when $\mathbf{s} = \mathbf{s}[w]$, it follows that $P = \{0\}$ and there is no gauge freedom. However, if we relax the requirement in (4.3), and merely separate (4.4) into its constituent parts according to $\mathbf{s}[w]$-grade, a nontrivial solution for S is revealed. It is easier to work with the adjoint group action and consider

$$\mathcal{L}_{\bar{q}} = K \mathcal{L}_q K^{-1}, \tag{4.5}$$

where

$$K = \exp(S) = I + \sum_{l>0} X_l, \tag{4.6}$$

I being the group identity and $X_l \in \bar{B}$ being of s$[w]$-grade $-l$ as well as homogeneous with respect to the differential grading on $\bar{B}$.[2]

4.3.1 Case I: Order Preserved on Lifting

Let Λ be of minimal positive grade and let the order be preserved on lifting, so that $N = n$ in the notation already adopted. Assuming, as usual, that w admits a regular eigenvector, Proposition 4.2.1 guarantees that the minimal positive degree which we may assign Λ is $j = 1$. Regularity ensures that

$$\mathcal{H}[w] = \text{Ker ad } \Lambda$$

and $\mathfrak{g}^{(1)}$ may be decomposed as the direct sum

$$\mathcal{H}[w] \oplus \mathcal{H}^{\perp}[w],$$

where $\mathcal{H}^{\perp}[w] := \text{Im ad } \Lambda$. Henceforth, we shall refer to the components of $Z \in \mathfrak{g}^{(1)}$ as $Z^{\parallel} \in \mathcal{H}[w]$ and $Z^{\perp} \in \mathcal{H}^{\perp}[w]$. Recall that $\mathcal{H}[w]$ is Abelian and note that

$$\left[\mathcal{H}[w], \mathcal{H}^{\perp}[w]\right] \subset \mathcal{H}^{\perp}[w],$$

a straightforward consequence of the Jacobi identity. Moreover, ad Λ restricted to $\mathcal{H}^{\perp}[w]$ is an isomorphism, as $\mathcal{H}[w] \cap \mathcal{H}^{\perp}[w] = \{0\}$.

Appealing to Proposition 4.1.4, it follows that q (and therefore $\bar{q}$) is of 0 degree and lies in $\mathcal{H}^{\perp}[w]$. We now solve for the components of K by looking at

$$\mathcal{L}_{\bar{q}} K = K \mathcal{L}_q,$$

which leads to[3]

$$D_x K + \bar{q} K - K q + \Lambda K - K \Lambda = 0. \tag{4.7}$$

[2] This is valid when working with the matrix representation, as if $K = \exp(S)$ has nonzero trace, we may rescale it to have trace $n+1 = \text{tr } I$, without affecting the adjoint action, so that $K = I + X$, where X is trace free and of negative s$[w]$-grade.

[3] This only makes sense because we are working with matrix representations of the algebra and its group.

Extracting components of homogeneous $\mathrm{s}[w]$-grade, we find that:

$$\bar{q} - q + [\Lambda, X_1] = 0 \tag{4.8}$$

$$D_x X_l + \bar{q} X_l - X_l q + [\Lambda, X_{l+1}] = 0 \quad l \geq 1 \tag{4.9}$$

The latter equation is the component of $\mathrm{s}[w]$-degree $-l$, $l \geq 1$ and the former of $\mathrm{s}[w]$-degree 0. This set of equations may be recursively solved for the X_l by breaking them up into their $\mathcal{H}[w]$ and $\mathcal{H}^{\perp}[w]$ projections.

Since $\bar{q}, q \in \mathcal{H}^{\perp}[w]$, (4.8) lies entirely in $\mathcal{H}^{\perp}[w]$. Thus,

$$X_1^{\perp} = -\mathrm{ad}\ \Lambda\ |_{\mathcal{H}^{\perp}[w]}^{-1} (\bar{q} - q),$$

and $X_1^{\perp}$ is homogeneous of differential degree 1 (the same as that of $q, \bar{q}$). The component of $\mathrm{s}[w]$-degree -1 decomposes as:

$$\begin{aligned} \| &: D_x X_1^{\|} + (\bar{q} X_1^{\perp} - X_1^{\perp} q)^{\|} = 0 \\ \perp &: D_x X_1^{\perp} + (\bar{q} X_1 - X_1 q)^{\perp} + [\Lambda, X_2] = 0 \end{aligned}$$

The first of these equations may be solved for $X_1^{\|}$ by integrating and setting the constant of integration to zero in order to maintain the aforementioned homogeneity of differential degree. We observe that the $\|$-component equation is homogeneous of differential degree 2 provided $X_1^{\|}$, and hence X_1, is of differential degree 1. The second equation now yields $X_2^{\perp}$, once again exploiting the isomorphism ad Λ when restricted to $\mathcal{H}^{\perp}[w]$. The decomposition of (4.9) for general l is:

$$\| : D_x X_l^{\|} + (\bar{q} X_l^{\perp} - X_l^{\perp} q)^{\|} = 0 \tag{4.10}$$

$$\perp : D_x X_l^{\perp} + (\bar{q} X_l - X_l q)^{\perp} + [\Lambda, X_{l+1}] = 0 \tag{4.11}$$

It follows that at each step the equations may be solved for $X_l^{\|}$ (from the kernel component) and $X_{l+1}^{\perp}$ (from the image component) in exactly the same way as has just been seen for $X_1^{\|}$ and $X_2^{\perp}$. Thus, we obtain a unique solution for K. Furthermore, each X_l is homogeneous of differential degree l. In practice, we shall only concern ourselves with those cases where the integration required does not lead to any nonlocal variables. We have therefore established:

Proposition 4.3.1 *Let $w \in W(\mathfrak{g})$ for which the order is preserved on lifting to $Aut(\mathfrak{g})$ and let $\Lambda \in \mathcal{H}_1[w]$. Then*

$$\mathcal{L}_{\bar{q}} = K\mathcal{L}_q K^{-1}$$

has a unique solution

$$K = I + \sum_{l>0} X_l,$$

where X_l is of $\mathfrak{s}[w]$-degree $-l$ and is homogeneous of degree l with respect to the differential grading.

Of course, K itself must satisfy an evolution equation implied by the zero-curvature equations

$$[\mathcal{L}_q, D_t + a + \Lambda'] = 0,$$

where $\Lambda' \in \mathcal{H}_k[w]$ and $a \in \mathfrak{g}^{(1)}_{\geq 0}(\mathfrak{s}[w]) \cap \mathfrak{g}^{(1)}_{<k}(\mathfrak{s}[w])$. Gauge transforming the zero-curvature equations under Ad K leads to

$$[\mathcal{L}_{\bar{q}}, D_t - D_t K \cdot K^{-1} + K(a + \Lambda')K^{-1}] = 0.$$

On the other hand, the zero-curvature equations for the $\bar{q}$ system are of the form

$$[\mathcal{L}_{\bar{q}}, D_t + \bar{a} + \Lambda'] = 0.$$

Thus,

$$[\mathcal{L}_{\bar{q}}, -D_t K \cdot K^{-1} + K(a + \Lambda')K^{-1} - (\bar{a} + \Lambda')] = 0.$$

Now, as $K = \exp(S)$ with $S \in C^\infty(\mathbb{R}, \mathfrak{g}^{(1)}_{<0})$, it follows that

$$-D_t K \cdot K^{-1} + K(a + \Lambda')K^{-1} - (\bar{a} + \Lambda') = \sum_{l=-\infty}^{k-1} Y_l,$$

where Y_l has $\mathfrak{s}[w]$-degree l. As will become apparent shortly, in order to preserve homogeneity with respect to the differential grading, Y_l must be homogeneous of differential degree $k - l$. Separating $[\mathcal{L}_{\bar{q}}, \sum Y_l] = 0$ into its constituents of $\mathfrak{s}[w]$-grade, yields:

$$\begin{aligned} D_x Y_l + [\bar{q}, Y_l] + [\Lambda, Y_{l-1}] &= 0 \quad l \leq k-1 \\ [\Lambda, Y_{k-1}] &= 0 \end{aligned}$$

Thus, $Y_{k-1} \in \mathcal{H}[w]$ so that splitting the equation for $l = k-1$ into its kernel and image components, we find:

$$\begin{aligned} \| &: D_x Y_{k-1} = 0 \\ \perp &: [\bar{q}, Y_{k-1}] + [\Lambda, Y_{k-2}] = 0 \end{aligned}$$

The first of these implies that $Y_{k-1} = 0$ in order to maintain homogeneity of differential degree, so that the second equation simply becomes $[\Lambda, Y_{k-2}] = 0$ whence $Y_{k-2} \in \mathcal{H}[w]$. Continuing this procedure for $l \leq k-1$ shows that $Y_l = 0$ for all $l \leq k-1$. Homogeneity of differential degree ensures that the Y_l moves up one differential degree each time l is decreased. Consequently, we find that K satisfies the evolution equation

$$D_t K = Ka - \bar{a}K + [K, \Lambda']. \tag{4.12}$$

Extracting each of the s$[w]$-grade constituents yields:

$$\begin{aligned} 0 &= \sum_{m=l}^{k-1} (X_{m-l} a_m - \bar{a}_m X_{m-l}) + [X_{k-l}, \Lambda'] \quad l = 0, \ldots, k-1 \quad &(4.13) \\ D_t X_l &= \sum_{m=0}^{k-1} (X_{m+l} a_m - \bar{a}_m X_{m+l}) + [X_{k+l}, \Lambda'] \quad l > 0 \quad &(4.14) \end{aligned}$$

where $X_0 = I$, $a = \sum_{l=0}^{k-1} a_l$ with a_l homogeneous of s$[w]$-grade l, and similarly for $\bar{a}$. This means that the evolution of X_l is determined by $X_l, \ldots, X_{l+k}$.

We shall impose suitable conditions on the X_l so that the above infinite system of evolution equations collapses to a finite system in such a way as to remain consistent with the original zero-curvature system, or in other words, so that the integrability conditions are preserved. In fact, such a condition is that $X_l = 0$ for some $l > 0$. Referring to (4.8) and (4.9) it is easily seen that if $X_l = 0$ for some l, then $X_{l+1} \in \mathcal{H}[w]$ whence $D_x X_{l+1} = 0$ so that $X_{l+1} = 0$. Recursively, it follows that $X_m = 0, \forall m \geq l$. A quick inspection of (4.14) reveals that the condition $X_l = 0$ is then consistent with the original integrability conditions. Of course, setting $X_1 = 0$ means $\bar{q} = q$ and the gauge transformation is just the identity. Therefore:

Proposition 4.3.2 *The condition $X_l = 0$ for arbitrary $l > 0$ is consistent with the integrability conditions in (4.2) in the case of those Heisenberg subalgebras, $\mathcal{H}[w]$, for which the order of w is preserved on lifting to $Aut(\mathfrak{g})$.*

4.3.2 Case II: Order Doubles on Lifting

We now consider the case where the order doubles upon lifting, so that $N = 2n$. Again, we assume that w admits a regular eigenvector. Proposition 4.2.1 tells us that the minimal positive degree for an element of $\mathcal{H}[w]$ is $j = 2$. Accordingly, we take Λ to be of degree 2, so that

$$\mathcal{L}_q = D_x + q_0 + q_1 + \Lambda,$$

where $q = q_0 + q_1$ expresses $q \in C^\infty(\mathbb{R}, Q)$, with $Q = \mathfrak{g}_0^{(1)}(\mathsf{s}[w]) \oplus \mathfrak{g}_1^{(1)}(\mathsf{s}[w])$, as a sum of its constituent parts of homogeneous $\mathsf{s}[w]$-grade. It follows from Proposition 4.2.1 that all elements homogeneous of odd $\mathsf{s}[w]$-degree lie in $\mathcal{H}^\perp[w]$. In particular, $q_1 \in \mathcal{H}^\perp[w]$. In the same way as before, we impose a differential grading and work in the algebra $\bar{B}$ involving $q, \bar{q}$ and their x-derivatives as well as nonlocal variables, if necessary. Analysis of the gauge transformation (4.5), with K defined as in (4.6), leads to the same system (4.7), namely,

$$D_x K + \bar{q}K - Kq + \Lambda K - K\Lambda = 0,$$

which, when separated into its components of homogeneous $\mathsf{s}[w]$, yields:

$$\begin{aligned} \bar{q}_1 - q_1 + [\Lambda, X_1] &= 0 && (4.15) \\ \bar{q}_0 - q_0 + \bar{q}_1 X_1 - X_1 q_1 + [\Lambda, X_2] &= 0 && (4.16) \\ D_x X_l + \bar{q}_0 X_l - X_l q_0 + \bar{q}_1 X_{l+1} - X_{l+1} q_1 + [\Lambda, X_{l+2}] &= 0 \quad l \geq 1 && (4.17) \end{aligned}$$

As usual, we split these equations up into their kernel and image parts. As (4.16) lies entirely in $\mathcal{H}^\perp[w]$, it completely determines X_1. From here on, things become somewhat problematic. Decomposing (4.16) into its kernel and image parts gives:

$$\parallel \ : \ (\bar{q}_0 - q_0 + \bar{q}_1 X_1 - X_1 q_1)^{\parallel} = 0 \tag{4.18}$$

$$\perp \ : \ (\bar{q}_0 - q_0 + \bar{q}_1 X_1 - X_1 q_1)^{\perp} + [\Lambda, X_2] = 0 \tag{4.19}$$

The latter of these equations may be solved for $X_2^{\perp}$, but the former appears to impose some extra conditions on X_1, which supposedly has already been found. Perhaps both this equation and the expression for X_1 obtained from (4.15) only admit the trivial solution $X_1 = 0$, a possibility to be discussed shortly. The rest of the equations in (4.17) are of $\mathsf{s}[w]$-degree l, and decompose into kernel and image parts if l is even, and just an image part if l is odd. Typically, then, the decomposition is, for $l \geq 1$:

$$\begin{aligned}
\bar{q}_1 X_{2l}^{\parallel} - X_{2l}^{\parallel} q_1 + [\Lambda, X_{2l+1}] &= -D_x X_{2l-1} - \bar{q}_0 X_{2l-1} + X_{2l-1} q_0 \\
&\quad - \bar{q}_1 X_{2l}^{\perp} + X_{2l}^{\perp} q_1 && (4.20) \\
D_x X_{2l}^{\parallel} + (\bar{q}_1 X_{2l+1} - X_{2l+1} q_1)^{\parallel} &= -(\bar{q}_0 X_{2l}^{\perp} - X_{2l}^{\perp} q_0)^{\parallel} && (4.21) \\
[\Lambda, X_{2l+2}^{\perp}] &= -D_x X_{2l}^{\perp} - (\bar{q}_1 X_{2l+1} - X_{2l+1} q_1)^{\perp} \\
&\quad - (\bar{q}_0 X_{2l} - X_{2l} q_0)^{\perp} && (4.22)
\end{aligned}$$

The left hand sides contain the unknown components of the X_m at each step, while the right hand sides are completely determined at that particular step. The first two equations must be solved simultaneously to obtain $X_{2l}^{\parallel}$, and therefore all of X_{2l}, as well as $X_{2l+1} = X_{2l+1}^{\perp}$. The last equation may then be solved for $X_{2l+2}^{\perp}$ and so on recursively. Comparison of these equations with (4.10), (4.11), for the analogous case when w does not double in order, illustrates that the present situation is far more complicated. Furthermore, it is not clear how to specify the X_l so that each of Equations (4.15)–(4.17) is homogeneous with respect to the differential grading. Although (4.15) suggests that X_1 is of the same differential degree as $q_1, \bar{q}_1$, namely 1, then (4.16) would mean that not only X_2 but $q_0, \bar{q}_0$ have differential degree 2. Moreover, (4.17) presents problems in that $D_x X_l$ has differential degree l, while X_{l+2} has differential degree $l+2$. Of course, the crux of the matter is whether or not it is feasible to expect the X_l in $K = I + \sum_{l>0} X_l$ to be both homogeneous of s[w]-degree $-l$ *and* homogeneous in the differential grading at the same time. The preceding remarks would suggest that such an assumption is unwarranted. Yet another reason for abandoning this approach is that $[\mathcal{L}_{\bar{q}}, \sum Y_l] = 0$ does not necessarily mean that all the Y_l are zero, a fact that was crucial in determining the evolution of K for the case when w did not double in order.

It could be possible to impose some sort of *Ansatz* that renders the equations for the X_l more manageable. It was mentioned that $X_1 = 0$ might help to reconcile (4.16) and (4.18); in fact, it would result in $\bar{q}_1 = q_1$ and $\bar{q}_0^{\|} = q_0^{\|}$. From this it follows that

$$X_2^{\perp} = -\mathrm{ad}\,\Lambda\,|_{\mathcal{H}^{\perp}[w]}^{-1}(\bar{q}_0 - q_0),$$

and, for $l \geq 1$ the same system of equations as (4.20)–(4.22), with $X_1 = 0$, $\bar{q}_1 = q_1$ and $\bar{q}_0^{\|} = q_0^{\|}$. The resulting system is still unsatisfactory in terms of complexity and issues of differential grade. A further simplification is requiring that $\bar{q}_1 = q_1 = 0$, though there is the risk that this goes too far and renders any resulting system of evolution equations entirely trivial. However, pursuing this idea, we find that in in addition to $X_1 = 0$, $\bar{q}_0^{\|} = q_0^{\|}$ and $X_2^{\perp}$ as above, we have $[\Lambda, X_3] = 0$ from (4.17), so that $X_3 = 0$ (recall that $\mathcal{H}[w]$ consists only of elements of even degree). Next,

$$D_x X_2^{\|} + (\bar{q}_0^{\perp} X_2^{\perp} - X_2^{\perp} q_0^{\perp})^{\|} = 0,$$

which may be integrated to find $X_2^{\|}$ and hence all of X_2. The image component of the same equation gives

$$D_x X_2^{\perp} + (\bar{q}_0 X_2 - X_2 q_0)^{\perp} + [\Lambda, X_4] = 0,$$

which allows the solution of $X_4^{\perp}$. The next equation is then just $[\Lambda, X_5] = 0$, so that $X_5 = 0$. Inductively, it follows that $X_l = 0$ for all odd l so that

$$K = I + \sum_{l>0} X_{2l},$$

where X_{2l} is homogeneous of s$[w]$-degree $-2l$. Stipulating that X_{2l} be homogeneous in the differential grading of degree l is then consistent with the equations

$$D_x X_{2l} + \bar{q}_0 X_{2l} - X_{2l} q_0 + [\Lambda, X_{2l+2}] = 0, \quad l \geq 1, \tag{4.23}$$

that determine the X_{2l}.

A similar analysis to that performed in the case when w does not double in order shows that

$$D_t K = Ka - \bar{a}K + [K, \Lambda'],$$

and that the evolution of X_{2l} is determined by its successors X_{2m}, for $m \geq l$. Moreover, if $X_{2l} = 0$ for some l, then the usual kernel and image decomposition on (4.23) shows that $X_{2m} = 0$, $\forall m \geq l$. Thus, we have an analogue of Proposition (4.3.2):

Proposition 4.3.3 *Let $w \in W(\mathfrak{a}_n)$ for which the order doubles on lifting to $Aut(\mathfrak{a}_n)$. If we stipulate that $\bar{q}_1 = q_1 = 0$ for $\mathcal{L}_{\bar{q}} = K\mathcal{L}_q K^{-1}$, where*

$$K = I + \sum_{l>0} X_{2l},$$

with X_{2l} homogeneous of s$[w]$*-degree $-2l$, then the condition $X_{2l} = 0$ for arbitrary $l > 0$ is consistent with the integrability conditions in (4.2).*

In conclusion, it is not at all clear how to carry over the analysis of Subsection 4.3.1 to the case when the order of w doubles on lifting, without imposing some sort of *Ansatz*. Given this unsatisfactory situation it is proposed to postpone investigation of such matters until a later time, and concentrate in this thesis on the case when the order of w is preserved.

Chapter 5

A Class of Automorphisms Preserving Gradation

In this chapter, the generalization of Guil's automorphism τ in [25] is constructed. In §5.1, so-called extended Carter diagrams are introduced for $\mathfrak{a}_n^{(1)}$ and the appropriate generalization of Guil's τ, also denoted here by τ, is proved to be a *bona fide* root space automorphism. The lifts of such a general τ and the Weyl group element w to automorphisms of $\mathfrak{a}_n^{(1)}$ are constructed in §5.2. In §5.3, a detailed examination of the lifts, denoted $\tilde{\tau}$ and $\tilde{w}$, is presented with particular reference to their behaviour with respect to the $\mathrm{s}[w]$-gradation induced by w. This is the content of Theorem 5.3.4 and Corollaries 5.3.5, 5.3.6 and 5.3.7. In particular, the first and third of these corollaries show that $\tilde{\tau}$ and a variant $\tilde{\tau}'$ both preserve the $\mathrm{s}[w]$-gradation.

5.1 Extended Carter Diagrams and Involutions

In [25], Guil defines an involution on $\mathfrak{a}_n^{(1)}$ (*i.e.* an automorphism of order 2) that preserves the principal Heisenberg subalgebra as well as the principal gradation of $\mathfrak{a}_n^{(1)}$. He constructed this involution, which he called τ, from a symmetry of the extended Dynkin diagram of $\mathfrak{a}_n$. As this diagram is just a regular $(n+1)$-gon, its symmetry group is the dihedral group, D_{n+1}. The automorphism τ is obtained by reflecting in the line through a given vertex and its opposite vertex (if n is odd) or opposite edge midpoint (if n is even), as in Figure 5.1. Thus, τ has order 2. By disregarding the given vertex, which is fixed by such a reflection, it follows that the restriction of τ is just the diagram symmetry for $\mathfrak{a}_n$.

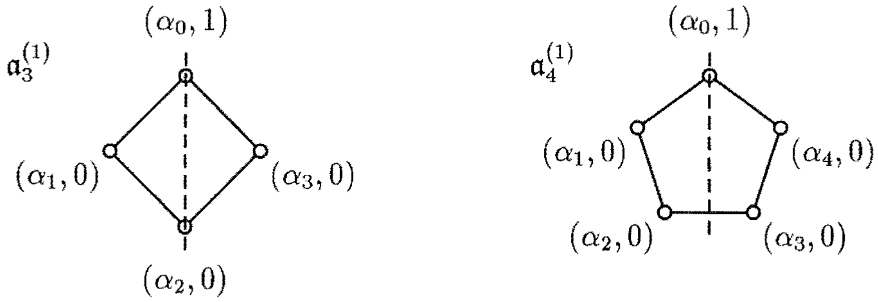

Figure 5.1: Reflectional symmetry of the extended Dynkin diagrams of $\mathfrak{a}_n^{(1)}$

In the case when n is odd, there is another choice for τ, namely the reflection in the line through midpoints of opposite edges, as shown in Figure 5.2. However, this involution, τ', is expressible in terms of the previous τ and the Coxeter element w_C, by $\tau' = \tau w_C$.

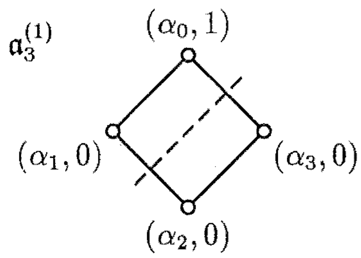

Figure 5.2: Alternative symmetry of the extended Dynkin diagram of $\mathfrak{a}_n^{(1)}$, n odd

This relationship is easily understood by considering the effect of w_C on the extended diagram:

$$w_C = r_{\alpha_1} r_{\alpha_2} \ldots r_{\alpha_n} : \alpha_k \mapsto \alpha_{k+1}.$$

Hence, w_C corresponds to a rotation through the angle $\frac{2\pi}{n+1}$ about the centre of the diagram, taking each vertex to its immediate successor. Note that the order of such a rotation is $n+1$ in agreement with the order of w_C, the Coxeter number of $\mathfrak{a}_n$.

It follows that it is enough to initially consider the diagram automorphism of $\mathfrak{a}_n$, whence τ is found by lifting to $\mathfrak{a}_n^{(1)}$ and for n odd, the second kind of involution is simply τw_C. The importance of these involutions is that certain equations of the Drinfel'd-Sokolov hierarchies over the principal Heisenberg subalgebra are preserved by the "gauge symmetries" taking the typical system with dependent variables q to that with $\bar{q} = \tau(q)$. Specialization of these leads to new zero-curvature systems over the subalgebra of fixed points of τ. In this section, it is shown how to generalize the involution τ for arbitrary Heisenberg subalgebras, *i.e.* those pertaining to the conjugacy classes of $W(\mathfrak{a}_n)$ other than the Coxeter class. Later, we shall construct specializations of Drinfel'd-Sokolov hierarchies over the fixed point subalgebras of such τ.

We recall the representatives of the conjugacy classes of $W(\mathfrak{a}_n)$ and the direct sum decomposition they induce on $\mathfrak{h}^*$, the dual of the CSA, with respect to the inner product defined by the Killing form. The conjugacy classes are in one to one correspondence with the partitions of the integer $n+1$. A typical representative of a given conjugacy class is

$$w = w_1 w_2 \dots w_r,$$

with associated regular subalgebra

$$\mathfrak{a}_{n_1} \oplus \mathfrak{a}_{n_2} \oplus \cdots \oplus \mathfrak{a}_{n_r} \subset \mathfrak{a}_n,$$

such that $\sum_{k=1}^{r}(n_k+1) = n+1$, $n_1 \geq n_2 \geq \ldots \geq n_r \geq 0$, with $\mathfrak{a}_0 := \{0\}$, and w_k is the Coxeter element of $\mathfrak{a}_{n_k}$. As the $\mathfrak{a}_{n_k}$ are mutually orthogonal, the Carter diagram is the sequence of (unjoined) Dynkin diagrams of the $\mathfrak{a}_{n_k}$.

For want of a better name, we shall define the *extended* Carter diagram associated with $w \in W(\mathfrak{a}_n)$, to be the disconnected sequence of extended Dynkin diagrams of $\mathfrak{a}_{n_k}^{(1)}$, along with the "missing" nodes corresponding to those simple roots whose reflections r_α do not appear in the expression $w = w_1 w_2 \dots w_r$. These nodes are to be inserted in the appropriate places between the extended Dynkin diagrams of $\mathfrak{a}_{n_k}^{(1)}$ so that along the base line of the overall diagram the nodes $\alpha_1, \dots, \alpha_n$ appear in order.

Along with the regular semisimple subalgebra decomposition, we have the corresponding direct sum decomposition of $\mathfrak{h}^*$,

$$\mathfrak{h}^* = \mathfrak{h}_1^* \oplus \mathfrak{h}_2^* \oplus \cdots \oplus \mathfrak{h}_r^* \oplus \mathfrak{h}'^*,$$

and, of course, the associated decomposition of $\mathfrak{h}$ itself, due to the canonical isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$ induced by the Killing form. As $w\mid_{\mathfrak{a}_{n_k}} = w_k = w_C(\mathfrak{a}_{n_k})$, and $\det(\mathrm{id} - w_C(\mathfrak{a}_{n_k})) \neq 0$, it follows that $\mathfrak{h}'^*$ contains all the fixed points of w.[1] That $\mathfrak{h}'^*$ is the subspace comprising *only* the fixed points of w is evident from an easy dimensionality consideration:

$$\dim \mathfrak{h}_{n_k} = \mathrm{rank}\ \mathfrak{a}_{n_k} = n_k \geq 0$$

so that

$$\dim \mathfrak{h}'^* = \dim \mathfrak{h}^* - \sum \dim \mathfrak{h}_{n_k} = n - \sum n_k.$$

Now, each w_k, being the Coxeter element of $\mathfrak{a}_{n_k}$, may be written as the product of n_k distinct reflections in simple roots of $\mathfrak{a}_{n_k}$, so that $w = w_1 w_2 \ldots w_r$ is the product of $\sum n_k$ distinct simple reflections in $W(\mathfrak{a}_n)$. A basis for the subspace of fixed points of w is neatly described in terms of the fundamental dominant weights λ_j, $j = 1, \ldots, n$, of $\mathfrak{a}_n$, with respect to a given basis of simple roots. These were defined in §3.3 to be a dual basis with respect to the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}^*$:

$$2\frac{\langle \lambda_j, \alpha_k \rangle}{\langle \alpha_k, \alpha_k \rangle} = \delta_{jk}.$$

For $\mathfrak{a}_n$, we previously quoted the formula [33]

$$\lambda_j = \frac{1}{n+1}\left[(n-j+1)\sum_{k=1}^{j} k\alpha_k + j\sum_{k=j+1}^{n}(n-k+1)\alpha_k\right].$$

Since $\langle \alpha_k, \alpha_k \rangle = 2$, the simple reflections r_{α_k} are defined by

$$r_{\alpha_k}(\beta) = \beta - \langle \beta, \alpha_k \rangle \alpha_k, \ \forall \beta \in \mathfrak{h}^*,$$

[1] Recall that w_k is a primitive element of $\mathfrak{a}_{n_k}$, meaning that $\det(\mathrm{id} - w_k) = \det A_{n_k}$, where A_{n_k} is the (nonsingular) Cartan matrix of $\mathfrak{a}_{n_k}$. In the case of algebras of type $\mathfrak{a}$, the Coxeter elements are the only primitive elements.

and thus

$$r_{\alpha_k}(\lambda_j) = \lambda_j - \delta_{jk}\alpha_k.$$

In particular, λ_j is fixed by each r_{α_k}, $k \neq j$. Thus, as $w = \prod_{k\in I} r_{\alpha_k}$, for some subset $I \subset \{1,\ldots,n\}$, we see that each λ_j, $j \notin I$, is a fixed point of w. There are exactly $n - \sum n_k$ such λ_j, therefore these form a basis for the $(n - \sum n_k)$ dimensional subspace $\mathfrak{h}'^*$, so that $\mathfrak{h}'^*$ consists of precisely the fixed points of w. Note that the λ_j are not elements of the root space $\Delta \subset \mathfrak{h}'^*$, and that there is no basis for $\mathfrak{h}'^*$ consisting of roots only. (The subspaces $\mathfrak{h}^*_{n_k}$ are, however, spanned by roots, namely those simple roots occurring in the product representation of w_k.)

The fact that $\mathfrak{h}'^*$ is orthogonal to each of the $\mathfrak{h}_{n_k}$ is readily apparent owing to the fact that w, regarded as a sequence of reflections, is an isometry of $\mathfrak{h}^*$, so that if we choose any eigenvector $\alpha \in \mathfrak{h}^*_{n_k}$ of w_k, say $w_k(\alpha) = \epsilon\alpha$, where ϵ is the primitive (n_k+1)th root of unity (w_k has order $n_k + 1$) and any $\beta \in \mathfrak{h}'^*$, then

$$\langle \alpha, \beta\rangle = \langle w(\alpha), w(\beta)\rangle = \epsilon\langle \alpha, \beta\rangle,$$

so that $\langle \alpha, \beta\rangle = 0$.

Now that the decomposition structure is understood, the involution τ that generalizes Guil's construction may be defined. Corresponding to the decomposition $w = w_1 \ldots w_r$, define the disjoint subsets $I_1, \ldots, I_r$, of $\{1,\ldots,n\}$, where $j_k := \sum_{i=1}^{k}(n_i + 1)$, $k = 1,\ldots,r$:

$$\begin{aligned}
I_1 &= \{1,\ldots,j_1 - 1\},\\
I_2 &= \{j_1 + 1,\ldots,j_2 - 1\},\\
&\;\vdots\\
I_k &= \{j_{k-1} + 1,\ldots,j_k - 1\},\\
&\;\vdots\\
I_r &= \{j_{r-1} + 1,\ldots,j_r - 1\}.
\end{aligned}$$

Thus, each set I_k has n_k elements. Here we are taking the $n_i > 0$ so that $j_r = \sum_{i=1}^{r}(n_i + 1) \leq n + 1$. In other words, we ignore the trivial factors $\mathfrak{a}_0$ in the regular subalgebra decomposition (which corresponds to omitting the tail end of 1's in the integer partition of $n+1$). For each factor $\mathfrak{a}_{n_k}$ of the regular subalgebra, we take as a basis of simple roots

$$\{\alpha_p : j_{k-1}+1 \leq p \leq j_k - 1\}.$$

The definition of the I_k ensures that the $\mathfrak{a}_{n_k}$ are mutually orthogonal. Now define τ on $\mathfrak{h}^*$, by

$$\tau\mid_{\mathfrak{h}^*_{n_k}} = \nu_k, \;\; k = 1, \ldots, r,$$

where ν_k is the Dynkin diagram symmetry of $\mathfrak{a}_{n_k}$, and

$$\tau\mid_{\mathfrak{h}'^*} = -\mathrm{id}\ .$$

This induces an invertible linear transformation of $\mathfrak{h}^*$, preserving the direct sum decomposition as well as being an isometry (each restriction is an isometry on the component factor). It is not obvious, however, that τ preserves the root space Δ, a crucial property in order to eventually extend τ to be an automorphism of the whole algebra $\mathfrak{a}_n$. The problem is that $\mathfrak{h}'^*$ has no basis consisting of *roots* alone and so it is not immediately clear that the α_{j_k} for $k = 1, \ldots, r$ are mapped onto roots. However, the following result shows that this is indeed the case.

Theorem 5.1.1 *Let $w = w_1 w_2 \ldots w_r \in W(\mathfrak{a}_n)$ with corresponding regular subalgebra*

$$\mathfrak{a}_{n_1} \oplus \mathfrak{a}_{n_2} \oplus \ldots \oplus \mathfrak{a}_{n_r},$$

such that

$$w_k = r_{\alpha_{j_{k-1}+1}} r_{\alpha_{j_{k-1}+2}} \cdots r_{\alpha_{j_k-1}},$$

where

$$j_p := \sum_{l=1}^{p} (n_l + 1)$$

and $\{\alpha_k : 1 \leq k \leq n\}$ constitutes the simple roots of $\mathfrak{a}_n$, so that w_k is the Coxeter element for $\mathfrak{a}_{n_k}$. Futhermore, let

$$\mathfrak{h} = \mathfrak{h}_{n_1} \oplus \mathfrak{h}_{n_2} \oplus \ldots \oplus \mathfrak{h}_{n_r} \oplus \mathfrak{h}'$$

be the orthogonal decomposition induced on the CSA on whose dual $W(\mathfrak{a}_n)$ acts. Denote the Dynkin diagram automorphism of $\mathfrak{a}_{n_k}$ by ν_k. Define the invertible linear transformations τ and ξ of $\mathfrak{h}^*$ by

$$\tau\mid_{\mathfrak{h}^*_{n_k}}=\nu_k,\ k=1,\ldots,r,\ \tau\mid_{\mathfrak{h}'^*}=-id,$$

$$\xi(\lambda_p)=\lambda_{\nu_1(p)}-\lambda_{j_1},\ p\in I_1:=\{1,\ldots,j_1-1\},$$

$$\xi(\lambda_p)=-\lambda_{j_{k-1}}+\lambda_{\nu_k(p)}-\lambda_{j_k},\ p\in I_k:=\{j_{k-1}+1,\ldots,j_k-1\},\ k=2,\ldots,r,$$

$$\xi(\lambda_p)=-\lambda_p,\ p\notin\cup_k I_k,$$

ξ being defined in terms of the fundamental dominant weights of $\mathfrak{a}_n$. Then, $\xi=\tau$, whence

$$\begin{aligned}\tau(\alpha_{j_k}) &= -\sum_{p=j_{k-1}+1}^{j_{k+1}-1}\alpha_p\ \in\Delta,\ k=1,\ldots,r-1,\\ \tau(\alpha_{j_r}) &= -\sum_{p=j_{r-1}+1}^{j_r}\alpha_p\ \in\Delta.\end{aligned}$$

For the remaining $p\notin\cup_k I_k$, that is $p>j_r$,

$$\tau(\alpha_p)=-\alpha_p.$$

In particular, τ preserves the root space, Δ.

PROOF: First, consider $\alpha_{p-1},\alpha_p,\alpha_{p+1}\in\mathfrak{h}^*_{n_k}$, that is $p-1,p,p+1\in I_k$. Recall that the Cartan matrix gives the change of basis matrix from $\{\lambda_j : j=1,\ldots,n\}$ to $\{\alpha_j : j=1,\ldots,n\}$, so that

$$\alpha_p=-\lambda_{p-1}+2\lambda_p-\lambda_{p+1}.$$

Thus,

$$\begin{aligned}
\xi(\alpha_p) &= -\xi(\lambda_{p-1}) + 2\xi(\lambda_p) - \xi(\lambda_{p+1}) \\
&= \lambda_{j_{k-1}} - \lambda_{\nu_k(p-1)} + \lambda_{j_k} \\
&\quad -2\lambda_{j_{k-1}} + 2\lambda_{\nu_k(p)} - 2\lambda_{j_k} \\
&\quad +\lambda_{j_{k-1}} - \lambda_{\nu_k(p+1)} + \lambda_{j_k} \\
&= -\lambda_{\nu_k(p)+1} + 2\lambda_{\nu_k(p)} - \lambda_{\nu_k(p)-1} \\
&= \alpha_{\nu_k(p)},
\end{aligned}$$

on observing that $\nu_k(p \pm 1) = \nu_k(p) \mp 1$.

Next, consider α_p for $p = j_{k-1} + 1$, for which $\nu_k(p) = j_k - 1$. Thus,

$$\begin{aligned}
\xi(\alpha_p) &= -\xi(\lambda_{j_{k-1}}) + 2\xi(\lambda_p) - \xi(\lambda_{p+1}) \\
&= \lambda_{j_{k-1}} + 2(-\lambda_{j_{k-1}} + \lambda_{\nu_k(p)} - \lambda_{j_k}) - (-\lambda_{j_{k-1}} + \lambda_{\nu_k(p+1)} - \lambda_{j_k}) \\
&= 2\lambda_{\nu_k(p)} - \lambda_{j_k} - \lambda_{\nu_k(p)-1} = \alpha_{\nu_k(p)},
\end{aligned}$$

since $j_k = \nu_k(p) + 1$.

Similarly, it may be shown that $\xi(\alpha_p) = \alpha_{\nu_k(p)}$ when $p = j_k - 1$. The same type of argument works for $\mathfrak{h}^*_{n_1}$, where the only difference is that $\xi(\lambda_p) = \lambda_{\nu_1(p)} - \lambda_{j_1}$. This confirms that ξ and τ have the same action on $\mathfrak{h}^*_{n_k}$, for all $k - 1, \dots, n$, that is to say $\xi(\alpha_p) = \tau(\alpha_p)$ for all $p \in \cup_{k=1}^r I_k$.

For the remaining $p \notin \cup_{k=1}^r I_k$ such that $p > j_r$, for which Sp $\{\lambda_p\} = \mathfrak{h}'^*$, we have $\xi(\lambda_p) = -\lambda_p = \tau(\lambda_p)$, so that τ and ξ agree on $\mathfrak{h}'^*$ and therefore on all of $\mathfrak{h}^*$ as required. Furthermore, for such p, α_p is orthogonal to $\{\alpha_k : k \in \cup_k I_k\}$, the set of all the simple roots whose reflections constitute w, and therefore lies in $\mathfrak{h}'^*$. Thus, $\tau(\alpha_p) = -\alpha_p$ for $p > j_r$, as claimed.

It remains to show that $\tau = \xi$ has the alleged action on $\alpha_{j_1}, \dots, \alpha_{j_r}$. To establish this we make use of the fact that $\tau(\lambda_{j_k}) = -\lambda_{j_k}$ and that [33]

$$\lambda_{j_k} = \frac{1}{n+1}\left[(n+1-j_k)\sum_{p=1}^{j_k} p\alpha_p + j_k \sum_{p=j_k+1}^{n} (n+1-p)\alpha_p\right].$$

Now,

$$\begin{aligned}
\tau(\alpha_{j_k}) &= -\tau(\lambda_{j_k-1}) + 2\tau(\lambda_{j_k}) - \tau(\lambda_{j_k+1}) \\
&= -(-\lambda_{j_{k-1}} + \lambda_{\nu_k(j_k-1)} - \lambda_{j_k}) - 2\lambda_{j_k} \\
&\quad -(-\lambda_{j_k} + \lambda_{\nu_{k+1}(j_k+1)} - \lambda_{j_{k+1}}) \\
&= \lambda_{j_{k-1}} - \lambda_{j_{k-1}+1} - \lambda_{j_{k+1}-1} + \lambda_{j_{k+1}},
\end{aligned}$$

since $\nu_k(j_k-1) = j_{k-1}+1$ and $\nu_{k+1}(j_k+1) = j_{k+1}-1$.

Hence,

$$\begin{aligned}
(n+1)\tau(\alpha_{j_k}) &= (n+1-j_{k-1})\sum_{p=1}^{j_{k-1}} p\alpha_p + j_{k-1}\sum_{p=j_{k-1}+1}^{n}(n+1-p)\alpha_p \\
&\quad -(n-j_{k-1})\sum_{p=1}^{j_{k-1}+1} p\alpha_p - (j_{k-1}+1)\sum_{p=j_{k-1}+2}^{n}(n+1-p)\alpha_p \\
&\quad -(n+2-j_{k+1})\sum_{p=1}^{j_{k+1}-1} p\alpha_p - (j_{k+1}-1)\sum_{p=j_{k+1}}^{n}(n+1-p)\alpha_p \\
&\quad +(n+1-j_{k+1})\sum_{p=1}^{j_{k+1}} p\alpha_p + j_{k+1}\sum_{p=j_{k+1}+1}^{n}(n+1-p)\alpha_p \\
&= 0\cdot\sum_{p=1}^{j_{k-1}} p\alpha_p - (n+1-j_{k-1})(j_{k-1}+1)\alpha_{j_{k-1}+1} - \sum_{p=j_{k-1}+2}^{j_{k+1}-1} p\alpha_p \\
&\quad +(n+1-j_{k+1})j_{k+1}\alpha_{j_{k+1}} + j_{k-1}(n-j_{k-1})\alpha_{j_{k-1}+1} \\
&\quad -\sum_{p=j_{k-1}+2}^{j_{k+1}-1}(n+1-p)\alpha_p - j_{k+1}(n+1-j_{k+1})\alpha_{j_{k+1}} \\
&\quad +0\cdot\sum_{p=j_{k+1}+1}^{n}(n+1-p)\alpha_p \\
&= -(n+1)\alpha_{j_{k-1}+1} - (n+1)\sum_{p=j_{k-1}+2}^{j_{k+1}-1}\alpha_p \\
&= -(n+1)\sum_{p=j_{k-1}+1}^{j_{k+1}-1}\alpha_p.
\end{aligned}$$

Thus,

$$\tau(\alpha_{j_k}) = -\sum_{p=j_{k-1}+1}^{j_{k+1}-1}\alpha_p.$$

$\square$

5.2 Lifting to the Loop Algebra

Thus far, we have constructed τ and w as elements of $\mathrm{Aut}(\Delta)$. However, the whole of the zero-curvature formulation takes place within the loop algebra $\mathfrak{a}_n^{(1)}$. Therefore, it is necessary to define suitable lifts of $\tilde{\tau}$, the extension of τ to $\mathrm{Aut}(\mathfrak{a}_n)$, and $\tilde{w}$ to $\mathrm{Aut}(\mathfrak{a}_n^{(1)})$. The requirements are that the lift of $\tilde{\tau}$ (which we shall continue to denote by $\tilde{\tau}$, and likewise for $\tilde{w}$) should preserve an element of minimal positive degree of $\mathcal{H}[w]$ and that $\tilde{w}$ act as the identity on $\mathcal{H}[w]$. It follows, for those w admitting a regular eigenvector, that $\tilde{\tau}$ preserves the decomposition

$$\mathfrak{a}_n^{(1)} = \mathcal{H}[w] \oplus \mathcal{H}^{\perp}[w].$$

Moreover, in the regular case, $\tilde{\tau}$ preserves the $\mathsf{s}[w]$-gradation, and in one of the two general subcases of regular type, w may be lifted so as to preserve the gradation also.

In order to get a feel for the behaviour of the lifts, we first review the case when w is the Coxeter element, as described in Guil [25]. Here $w_C : \alpha_k \mapsto \alpha_{k+1}$ (subscripts modulo $n+1$) lifts to the automorphism whose action on the set of simple roots of $\mathfrak{a}_n^{(1)}$,

$$\tilde{\Pi} = \{(\alpha_0, 1), (\alpha_1, 0), \ldots, (\alpha_n, 0)\},$$

is given by:

$$\begin{aligned}
(\alpha_0, 1) &\mapsto (\alpha_1, 0), \\
(\alpha_k, 0) &\mapsto (\alpha_{k+1}, 0), \quad 1 \le k \le n-1, \\
(\alpha_n, 0) &\mapsto (\alpha_0, 1).
\end{aligned}$$

This corresponds to a rotation through one node of the extended Dynkin diagram. An automorphism of $\mathfrak{a}_n^{(1)}$ is obtained by requiring that

$$x_{\tilde{\alpha}} \mapsto c_\alpha x_{w_C(\tilde{\alpha})}$$

for $x_{\tilde{\alpha}} \in L^{\tilde{\alpha}}$ where $\tilde{\alpha} = (\alpha_k, j) \in \tilde{\Pi}$. In order that $\tilde{w}_C$ be the identity on $\mathcal{H}[w_C]$, we take $c_\alpha = 1$ and $x_{\tilde{\alpha}} \in L^{(\alpha_k, j)}$ to be represented by $\lambda^j E_{k,k+1}$ (where the subscripts are modulo $n+1$). Hence,

$$\begin{aligned} &\lambda^{j+1}x_0 \mapsto \lambda^j x_1, && \lambda^{j-1}y_0 \mapsto \lambda^j y_1, \\ &\lambda^j x_k \mapsto \lambda^j x_{k+1}, && \lambda^j y_k \mapsto \lambda^j y_{k+1}, \quad 1 \le k \le n-1, \\ &\lambda^j x_n \mapsto \lambda^{j+1}x_0, && \lambda^j y_n \mapsto \lambda^{j-1}y_0, \end{aligned}$$

where $j \in \mathbb{Z}$ and, as usual, $x_k = E_{k,k+1}$, $y_k = E_{k-1,k}$. To see why this is so, first note that the action induced from the root map means the above holds for $j = 0$. Consider the action on y_0:

$$y_0 = [x_1, \ldots, x_n] \mapsto [x_2, \ldots, \lambda x_0] = \lambda y_1,$$

and similarly $x_0 = [y_n, \ldots, y_1] \mapsto \lambda^{-1}x_1$. Here the notation $[z_1, \ldots, z_k]$ is being used as shorthand for $\mathrm{ad}\ z_1 \circ \cdots \circ \mathrm{ad}\ z_{k-1}(z_k)$, as in [25, 28]. Moreover,

$$\lambda h_0 = [\lambda x_0, y_0] \mapsto [x_1, \lambda y_1] = \lambda h_1.$$

In similar vein, it follows inductively that for all $j \in \mathbb{Z}$, $\lambda^j \otimes \mathfrak{h} \mapsto \lambda^j \otimes w_C(\mathfrak{h})$ and, as claimed, $\lambda^{j+1}x_0 \mapsto \lambda^j x_1$ *etc.*

A consequence of this is that $\tilde{w}_C$ preserves the decomposition of the loop algebra into the principal Heisenberg subalgebra and its orthogonal complement as well as the principal gradation associated with $\mathbf{s}_C = (1, \ldots, 1)$. This follows by virtue of the fact that $\{\lambda x_0, x_1, \ldots, x_n, \lambda^{-1}y_0, y_1, \ldots, y_n\}$ generates $\mathfrak{a}_n^{(1)}$ and clearly $\tilde{w}_C$ acts as the identity on $\mathcal{H}[w_C]$ (recall that the projections of $\mathcal{H}[w_C]$ onto $\mathfrak{a}_n$ lie in the fixed point subspace of w_C). Regularity ensures that $\mathcal{H}[w_C]$ coincides with Ker ad Λ, for suitable $\Lambda \in \mathcal{H}[w_C]$, and so the automorphism $\tilde{w}_C$ is guaranteed to preserve $\mathcal{H}^{\perp}[w_C] =$ Im ad Λ. Furthermore, note that $\deg \lambda^{j+1}x_0 = \deg \lambda^j x_1 = \cdots = \deg \lambda^j x_n = j$. Consequently, $\tilde{w}_C$ preserves the principal gradation of $\mathfrak{a}_n^{(1)}$. The fact that $\tilde{w}_C$ preserves the cyclic order in commutators means that the general homogeneous element $\lambda^j z$, $z \in \mathfrak{a}_n$, gets mapped to $\lambda^{j'} w_C(z)$, where j' is the appropriate value to preserve degree, and there is no plus or minus sign change, in contrast to the action of $\tilde{\tau}$, as we shall soon see.

The corresponding automorphism τ on the root space of $\mathfrak{a}_n$ has already been determined for general w and is the same as that given by Guil [25] for the Coxeter element. In this case, $\tau = \nu$, the Dynkin diagram automorphism of $\mathfrak{a}_n$, and the lift to the root space of $\mathfrak{a}_n^{(1)}$ is determined by its action on $\tilde{\Pi}$:

$$\begin{aligned}(\alpha_0, 1) &\mapsto (\alpha_0, 1),\\ (\alpha_k, 0) &\mapsto (\alpha_{\nu(k)}, 0), \quad 1 \le k \le n.\end{aligned}$$

This corresponds to a reflection through the axis of symmetry passing through $(\alpha_0, 1)$ of the regular $(n+1)$-gon that comprises the extended Dynkin diagram. As with w_C, an automorphism $\tilde{\tau}$ of $\mathfrak{a}_n^{(1)}$ is obtained by deeming that

$$x_{\tilde{\alpha}} \mapsto x_{\tau(\tilde{\alpha})}$$

for $x_{\tilde{\alpha}} \in L^{\tilde{\alpha}}$ where $\tilde{\alpha} \in \tilde{\Pi}$. Hence,

$$\begin{aligned}\lambda x_0 &\mapsto \lambda x_0, &\qquad \lambda y_0 &\mapsto \lambda y_0,\\ x_k &\mapsto x_{\nu(k)}, &\qquad y_k &\mapsto y_{\nu(k)}.\end{aligned}$$

In this case,

$$\begin{aligned}y_0 = [x_1, \ldots, x_n] \mapsto [x_{\nu(1)}, \ldots, x_{\nu(n)}] &= [x_n, \ldots, x_1]\\ &= (-1)^{n-1}[x_1, \ldots, x_n]\\ &= (-1)^{n+1} y_0,\end{aligned}$$

where use has been made of the Jacobi identity to get the appropriate power of -1 when changing order in the commutator. Also,

$$\lambda h_0 = [\lambda x_0, y_0] \mapsto [\lambda x_0, (-1)^{n+1} y_0] = (-1)^{n+1} \lambda h_0.$$

It will be crucial in specializing to know the precise effect of $\tilde{\tau}$ on $\mathcal{H}[w_C]$. To this end, consider the action on $\lambda^m x_1$:

$$[h_0, x_1] = -x_1 \quad \Rightarrow \lambda^m x_1 = (-1)^m \mathrm{ad}^m(\lambda h_0)\, x_1,$$

so that

$$\begin{aligned}\lambda^m x_1 \overset{\tilde{\tau}}{\mapsto} (-1)^m \mathrm{ad}^m((-1)^{n+1}\lambda h_0)\; x_{\nu(1)} &= (-1)^{m(n+2)}\mathrm{ad}^m(\lambda h_0)\; x_n \\ &= (-1)^{mn}(-1)^m \lambda^m x_n \\ &= (-1)^{(n+1)m}\lambda^m x_n.\end{aligned}$$

Note that $(n+1)m = \deg(\lambda^m x_n) - 1$. From these observations, it is straightforward to show that

$$\tilde{\tau}(\lambda^m h_j) = (-1)^{(n+1)m}\lambda^m h_{\nu(j)} = (-1)^{\deg(\lambda^m h_j)}\lambda^m h_{\nu(j)},\; 0 \leq j \leq n,$$

and

$$\tilde{\tau}(\lambda^m z) = (-1)^{\deg(\lambda^m z)-1}\lambda^m \tilde{\tau}(z),$$

where $z \in \{\lambda x_0, x_1, \ldots, x_n, \lambda^{-1}y_0, y_1, \ldots, y_n\}$.

Once again, $\tilde{\tau}$ preserves the principal gradation and the decomposition of $\mathfrak{a}_n^{(1)}$ into $\mathcal{H}[w_C] \oplus \mathcal{H}^{\perp}[w_C]$, the latter assertion being a consequence of the fact that $\tilde{\tau}(\Lambda) = \Lambda$, where $\Lambda = \lambda x_0 + x_1 + \cdots + x_n$ is the minimal principal Heisenberg element of positive degree, namely 1, and is regular so that $\mathcal{H}[w_C] = \mathrm{Ker\ ad}\ \Lambda$. The lift $\tilde{\tau}$ has order 2.

To determine the action of $\tilde{\tau}$ on $\mathcal{H}[w_C]$, note that $\mathcal{H}[w_C]$ consists of *precisely* the fixed points of $\tilde{w}_C$, of which a typical element, of degree $(n+1)m + j$, $0 \leq j \leq n$, is

$$\begin{aligned}\Lambda_{m,j} \;:=\; & \lambda^m[x_1, \ldots, x_j] + \lambda^m[x_2, \ldots, x_{j+1}] + \cdots \\ & + \lambda^m[x_{n-j+1}, \ldots, x_n] + \lambda^m[x_{n-j+2}, \ldots, \lambda x_0] + \cdots \\ & + \lambda^m[\lambda x_0, \ldots, x_{j-1}].\end{aligned}$$

Since $\tilde{\tau}$ preserves both the principal gradation and $\mathcal{H}[w_C]$, it follows that $\tilde{\tau}(\Lambda_{m,j}) = \pm\Lambda_{m,j}$, as $\Lambda_{m,j}$ is the only Heisenberg element of degree $(n+1)m+j$, up to scalar multiples, and $\tilde{\tau}$ has order 2. Therefore, it is enough to check the action on $\lambda^m[x_1, \ldots, x_j]$ in order to determine the sign.

$$\begin{aligned}
\tilde{\tau}(\lambda^m[x_1,\ldots,x_j]) &= [\tilde{\tau}(\lambda^m x_1),\tilde{\tau}(x_2),\ldots,\tilde{\tau}(x_j)] \\
&= (-1)^{(n+1)m}[\lambda^m x_n, x_{n-1},\ldots,x_{\nu(j)}] \\
&= (-1)^{(n+1)m}(-1)^{j-1}\lambda^m[x_{\nu(j)},\ldots,x_n] \\
&= (-1)^{(n+1)m+j-1}\lambda^m\tilde{w}_C^{n-j}[x_1,\ldots,x_j].
\end{aligned}$$

Thus,

$$\tilde{\tau}(\Lambda_{m,j}) = (-1)^{\deg \Lambda_{m,j}-1}\Lambda_{m,j}.$$

Of course, associated with any vector space direct sum decomposition is a projection operator.

Proposition 5.2.1 *The projection operator* $\pi_{w_C} : \mathfrak{a}_n^{(1)} \to \mathcal{H}[w_C]$ *that determines the vector space direct sum decomposition*

$$\mathfrak{a}_n^{(1)} = \mathcal{H}[w_C] \oplus \mathcal{H}^{\perp}[w_C]$$

is given by

$$\pi_{w_C} = \frac{1}{n+1}\sum_{j=0}^{n}\tilde{w}_C^j.$$

Proof: As already noted, by construction, $\mathcal{H}[w_C]$ is the lift to $\mathfrak{a}_n^{(1)}$ of $\psi(\mathfrak{h})$ where

$$\sigma_{\mathbf{s}[w_C]} = \psi\tilde{w}_C\psi^{-1},$$

and $\psi(\mathfrak{h})$ consists of the fixed points of $\tilde{w}_C : \mathfrak{a}_n \to \mathfrak{a}_n$ in their entirety. On lifting, it follows that $\mathcal{H}[w_C]$ is the fixed point subspace of $\tilde{w}_C : \mathfrak{a}_n^{(1)} \to \mathfrak{a}_n^{(1)}$ (allowing for the slight abuse of notation in the two different roles of $\tilde{w}_C$). Therefore,

$$\pi_{w_C} \mid_{\mathcal{H}[w_C]} = \mathrm{id}.$$

To complete the proof, it is enough to show that $\mathcal{H}^{\perp}[w_C]$ is the span of the non unit eigenspaces of $\tilde{w}_C$, so that if $X \in \mathfrak{a}_n^{(1)}$ is an eigenvector of $\tilde{w}_C$ with eigenvalue $\epsilon \neq 1$ a primitive $(n+1)$-th root of unity,

$$(n+1)\pi_{w_C}(X) = (1+\epsilon+\epsilon^2+\cdots+\epsilon^n)X = 0.$$

As noted earlier, regularity ensures that $\tilde{w}_C$ preserves $\mathcal{H}^\perp[w_C]$, as then $\mathcal{H}^\perp[w_C] = \mathrm{Im}\ \mathrm{ad}\ \Lambda$, for some $\Lambda \in \mathcal{H}[w_C] = \mathrm{Ker}\ \mathrm{ad}\ \Lambda$, so that if $X = [\Lambda, Y] \in \mathcal{H}^\perp[w_C]$, then

$$\tilde{w}_C(X) = [\tilde{w}_C(\Lambda), \tilde{w}_C(Y)] = [\Lambda, \tilde{w}_C(Y)] \in \mathcal{H}^\perp[w_C].$$

Now consider any eigenvector X of $\tilde{w}_C$, with eigenvalue $\epsilon \neq 1$ some primitive $(n+1)$-th root of unity. The eigenvector X decomposes as

$$X = X^{\parallel} + X^{\perp},$$

where $X^{\parallel} \in \mathcal{H}[w_C]$ is a fixed point of $\tilde{w}_C$ and $X^\perp \in \mathcal{H}^\perp[w_C]$. Hence,

$$\tilde{w}_C X = \epsilon X \Rightarrow X^{\parallel} + \tilde{w}_C X^\perp = \epsilon X^{\parallel} + \epsilon X^\perp.$$

Since $\tilde{w}_C$ preserves the decomposition, it follows that $X^{\parallel} = 0$ and $\tilde{w}_C X^\perp = \epsilon X^\perp$. Thus, any non unit eigenvector lies in $\mathcal{H}^\perp[w_C]$ and dimensionality shows that the span of all such eigenvectors constitute $\mathcal{H}^\perp[w_C]$. □

The projection operator π_{w_C} provides a convenient way of generating $\mathcal{H}[w_C]$. It preserves the principal gradation as $\tilde{w}_C$ does, and so the fastest way to find a Heisenberg element of given degree is to take a typical element of that degree and apply π_{w_C} to it. By the same token, it is straightforward to list those elements annihilated by π_{w_C} and so obtain a description of $\mathcal{H}^\perp[w_C]$.

Proposition 5.2.2 *Consider the element $\lambda^m[x_1, \ldots, x_k]$, where $1 \leq k \leq n$, which has degree $(n+1)m + k$ in $\mathfrak{a}_n^{(1)}$. Then*

$$\begin{aligned}
(n+1)\pi_{w_C}\lambda^m[x_1, \ldots, x_k] &= \sum_{j=0}^{n} \tilde{w}_C^j \lambda^m[x_1, \ldots, x_k] \\
&= \lambda^m[x_1, \ldots, x_k] + \lambda^m[x_2, \ldots, x_{k+1}] \\
&\quad + \lambda^m[x_3, \ldots, x_{k+2}] + \cdots + \lambda^m[x_{n-k+1}, \ldots, x_n] \\
&\quad + \lambda^m[x_{n-k+2}, \ldots, \lambda x_0] + \lambda^m[x_{n-k+3}, \ldots, \lambda x_0, x_1] \\
&\quad + \cdots + \lambda^m[\lambda x_0, \ldots, x_{k-1}].
\end{aligned}$$

This gives the typical principal Heisenberg element of degree $(n+1)m+k$. Note that there is only one such element up to linear independence, as all possible elements of *the required degree appear in the expression for the Heisenberg element. This reflects the fact that w_C has one dimensional eigenspaces for any given eigenvector.*[2]

Furthermore, as π_{w_C} has the same action on any of the summands in the expression for $(n+1)\pi_{w_C}\lambda^m[x_1,\ldots,x_k]$, it follows that the elements of $\mathcal{H}^\perp[w_C]$ of degree $(n+1)m+k$, $1 \le k \le n$, are

$$\begin{aligned}
\lambda^m[\lambda x_0,\ldots,x_{k-1}] &- \lambda^m[x_1,\ldots,x_k],\\
\lambda^m[\lambda x_0,\ldots,x_{k-1}] &- \lambda^m[x_2,\ldots,x_{k+1}],\\
&\vdots\\
\lambda^m[\lambda x_0,\ldots,x_{k-1}] &- \lambda^m[x_{n-k+1},\ldots,x_n],\\
\lambda^m[\lambda x_0,\ldots,x_{k-1}] &- \lambda^m[x_{n-k+2},\ldots,\lambda x_0],\\
\lambda^m[\lambda x_0,\ldots,x_{k-1}] &- \lambda^m[x_{n-k+3},\ldots,\lambda x_0,x_1],\\
&\vdots\\
\lambda^m[\lambda x_0,\ldots,x_{k-1}] &- \lambda^m[x_n,\lambda x_0,\ldots,x_{k-2}].
\end{aligned}$$

Finally, all elements of degree $(n+1)m$, namely $\lambda^m \otimes \mathfrak{h}$, lie in $\mathcal{H}^\perp[w_C]$.

PROOF: Obvious. The fact that $\lambda^m \otimes \mathfrak{h} \subset \mathcal{H}^\perp[w_C]$ is a consequence of w_C having no fixed points in $\mathfrak{h}$. □

We now know the action of the automorphisms $\tilde{w}_C$ and $\tilde{\tau}$ on $\mathfrak{a}_n^{(1)}$. Furthermore, when n is odd, the behaviour of the second type of involution $\tilde{\tau}' = \tilde{\tau}\tilde{w}_C$ is also clear: it preserves the $\mathcal{H}[w_C] \oplus \mathcal{H}^\perp[w_C]$ decomposition as well as the principal gradation. Note that since $\tau' = \tau w_C = w_C^{-1}\tau$, as is readily apparent by considering the effects of w_C and τ on the Dynkin diagram of $\mathfrak{a}_n^{(1)}$, it also follows that

$$\tilde{\tau}' = \tilde{\tau}\tilde{w}_C = \tilde{w}_C^{-1}\tilde{\tau}.$$

Note also that $\tilde{\tau}$ and $\tilde{\tau}'$ have the same action on $\mathcal{H}[w_C]$ since the principal Heisenberg subalgebra is left pointwise fixed by $\tilde{w}_C$.

[2]In the standard matrix realization, the typical principal Heisenberg element corresponds to the so-called *circulant* matrices of the earlier literature, in particular [41].

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Of course, in general one could generate another involution from, say, $\bar{\tau}$, where $\bar{\tau}$ corresponds to the node map

$$j \mapsto l - j,$$

of the Dynkin diagram, where l is a fixed element of $\mathbb{Z}_{n+1}$. In the immediately previous discussion, τ corresponds to $l = 0$ and τ' to $l = n$. When n is even, the fixed point subalgebras of all the $\bar{\tau}$ are isomorphic, irrespective of l, and when n is odd, up to isomorphism the fixed point subalgebras of the $\bar{\tau}$ fall into two categories, namely those isomorphic to that of τ or τ'. This will be made more specific in due course.

Having investigated the behaviour of the automorphisms $\tilde{w}$ and $\tilde{\tau}$ for the Coxeter case, we proceed to the general situation. Now $w = w_1 w_2 \dots w_r$ where w_j is the Coxeter element for the simple factor $\mathfrak{a}_{n_j}$ of the corresponding regular subalgebra, $\oplus_{j=1}^{r} \mathfrak{a}_{n_j}$. First, we examine the lift of w to the whole loop algebra, this lift once again being denoted by $\tilde{w}$. Recall the notation of Theorem 5.1.1, where

$$w_k = r_{\alpha_{j_{k-1}+1}} r_{\alpha_{j_{k-1}+2}} \cdots r_{\alpha_{j_k - 1}},$$

and

$$j_p := \sum_{l=1}^{p} (n_l + 1).$$

Thus,

$$\begin{aligned}
w(\alpha_{j_1}) &= r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_{n_1}} r_{\alpha_{n_1+2}} (\alpha_{j_1}) \\
&= r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_{n_1}} (\alpha_{j_1} + \alpha_{j_1+1}) \\
&\ \ \vdots \\
&= \alpha_1 + \cdots \alpha_{j_1+1}.
\end{aligned}$$

Similarly,

$$w(\alpha_{j_k}) = \alpha_{j_{k-1}+1} + \cdots + \alpha_{j_k+1},$$

for $1 \le k \le r - 1$.

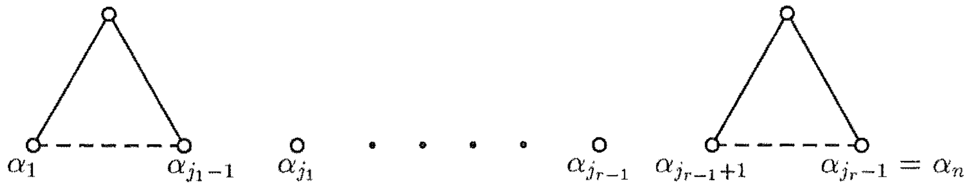

Figure 5.3: Extended Carter diagram for $j_r - 1 = n$

There remain two possibilities to consider: either $j_r - 1 = n$, as in Figure 5.3, in which case

$$\begin{aligned} w(\alpha_0) &= r_{\alpha_1} \cdots r_{\alpha_{j_{r-1}-1}} r_{\alpha_{j_{r-1}+1}} r_{\alpha_{j_{r-1}+2}} \cdots r_{\alpha_n}(\alpha_0) \\ &= r_{\alpha_1} \cdots r_{\alpha_{j_{r-1}-1}}(\alpha_{j_{r-1}+1} + \cdots + \alpha_n + \alpha_0) \\ &= \alpha_{j_{r-1}+1} + \cdots + \alpha_n + \alpha_0 + \alpha_1 \\ &= -(\alpha_2 + \cdots + \alpha_{j_{r-1}}), \end{aligned}$$

or $j_r - 1 < n$, as in Figure 5.4, whence

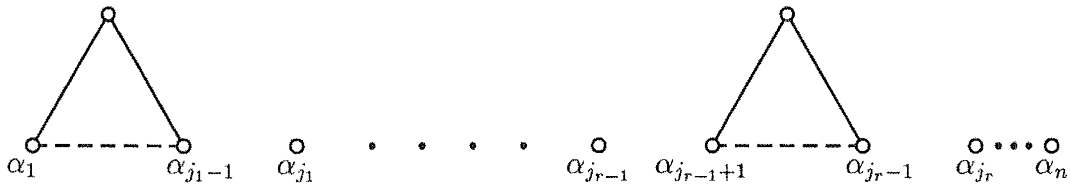

Figure 5.4: Extended Carter diagram for $j_r - 1 < n$

$$w(\alpha_0) = r_{\alpha_1}(\alpha_0) = \alpha_0 + \alpha_1 = -(\alpha_2 + \cdots + \alpha_n).$$

In this case,

$$w(\alpha_{j_r}) = \alpha_{j_{r-1}+1} + \cdots + \alpha_{j_r},$$

and for $k > j_r$, $w(\alpha_k) = \alpha_k$, as then $\alpha_k \in \mathfrak{h}'^*$.

In order that $\tilde{w}$ has the same action as the Coxeter lift on each of the factors $\mathfrak{a}_{n_k}^{(1)}$ of the loop algebra, the necessary action on the standard basis of simple roots for the loop algebra is as follows:

$$\begin{aligned}
(\alpha_{j_{k-1}+1},0) &\mapsto (\alpha_{j_{k-1}+2},0),\\
(\alpha_{j_{k-1}+2},0) &\mapsto (\alpha_{j_{k-1}+3},0),\\
&\vdots\\
(\alpha_{j_k-2},0) &\mapsto (\alpha_{j_k-1},0),\\
(\alpha_{j_k-1},0) &\mapsto (-\alpha_{j_{k-1}+1}-\cdots-\alpha_{j_k-1},1),
\end{aligned}$$

for $1 \leq k \leq r$, and

$$(\alpha_{j_k},0) \mapsto (\alpha_{j_{k-1}+1}+\cdots+\alpha_{j_k+1},-1),$$

for $1 \leq k \leq r-1$, taking $j_0 = 0$.

If $j_r - 1 = n$, then the description is completed by requiring that

$$(\alpha_0,1) \mapsto (-\alpha_2-\cdots-\alpha_{j_{r-1}},0).$$

On the other hand, if $j_r - 1 < n$, then

$$\begin{aligned}
(\alpha_{j_r},0) &\mapsto (\alpha_{j_{r-1}+1}+\cdots+\alpha_{j_r},-1),\\
(\alpha_k,0) &\mapsto (\alpha_k,0),\ k > j_r,\\
(\alpha_0,1) &\mapsto (-\alpha_2-\cdots-\alpha_n,1).
\end{aligned}$$

To see that this yields the Coxeter action on each of the subalgebras, we just consider the case for $j_r - 1 < n$ for a general factor $\mathfrak{a}^{(1)}_{n_k}$. The case for $j_r - 1 = n$ is similar. To confirm that $\tilde{w}$ has the desired action, it suffices to show that

$$\lambda[y_{j_k-1},\ldots,y_{j_{k-1}+1}] \mapsto x_{j_{k-1}+1}.$$

In order to determine the action, we describe $\lambda[y_{j_k-1},\ldots,y_{j_{k-1}+1}]$ in terms of commutators of elements lying in the root spaces of the simple roots:

$$\begin{aligned}
\lambda[y_{j_k-1},\ldots,y_{j_{k-1}+1}] &= [x_{j_k},x_{j_k+1},\ldots,x_{j_{k+1}-2},x_{j_{k+1}-1},\ldots,x_{j_{r-1}},x_{j_{r-1}+1},\ldots,\\
&\quad x_{j_r-2},x_{j_r-1},x_{j_r},x_{j_r+1},\ldots,x_n,\\
&\quad \lambda x_0,x_1,x_2,\ldots,x_{j_1-2},x_{j_1-1},\ldots,\\
&\quad x_{j_{k-2}},x_{j_{k-2}+1},\ldots,x_{j_{k-1}-2},x_{j_{k-1}-1},x_{j_{k-1}}]\\
&\overset{\tilde{w}}{\mapsto} [\lambda^{-1}[x_{j_{k-1}+1},\ldots,x_{j_k+1}],x_{j_k+2},\ldots,x_{j_{k+1}-1},\\
&\quad \lambda[y_{j_{k+1}-1},\ldots,y_{j_k+1}],\ldots,\lambda^{-1}[x_{j_{r-2}+1},\ldots,x_{j_{r-1}+1}],\\
&\quad x_{j_{r-1}+2},\ldots,x_{j_r-1},\lambda[y_{j_r-1},\ldots,y_{j_{r-1}+1}],\\
&\quad \lambda^{-1}[x_{j_{r-1}+1},\ldots,x_{j_r}],x_{j_r+1},\ldots,x_n,\lambda[y_n,\ldots,y_2],\\
&\quad x_2,x_3,\ldots,x_{j_1-1},\lambda[y_{j_1-1},\ldots,y_1],\ldots,\\
&\quad \lambda^{-1}[x_{j_{k-3}+1},\ldots,x_{j_{k-2}+1}],x_{j_{k-2}+2},\ldots,x_{j_{k-1}-1},\\
&\quad \lambda[y_{j_{k-1}-1},\ldots,y_{j_{k-2}+1}],\lambda^{-1}[x_{j_{k-2}+1},\ldots,x_{j_{k-1}+1}]]\\
&= [[x_{j_{k-1}+1},\ldots,x_{j_k}],\ldots,[x_{j_{r-2}+1},\ldots,x_{j_r-1}],\\
&\quad [x_{j_{r-1}+1},\ldots,x_n],[y_n,\ldots,y_2],y_1,\ldots,\\
&\quad [x_{j_{k-3}+1},\ldots,x_{j_{k-2}}],[x_{j_{k-2}+1},\ldots,x_{j_{k-1}+1}]]\\
&= [[x_{j_{k-1}+1},\ldots,x_n],[y_n,\ldots,y_1],[x_1,\ldots,x_{j_{k-1}+1}]]\\
&= [[y_{j_{k-1}},\ldots,y_1],[x_1,\ldots,x_{j_{k-1}+1}]]\\
&= x_{j_{k-1}+1}
\end{aligned}$$

Recall that in the standard matrix representation, $[x_{k_1},\ldots,x_{k_2}]$ is represented by E_{k_1,k_2+1} and $[y_{k_2},\ldots,y_{k_1}]$ by E_{k_2+1,k_1}, where $1\le k_1\le k_2\le n$. This is useful in performing the above sort of calculations and keeping track of factors of -1 arising from having to reverse the order in certain commutators.

Next, we examine the corresponding lift of τ to the loop algebra. The action of τ itself has been fully described in Theorem 5.1.1.

As with $\tilde{w}$, the effect on the simple roots of the loop algebra is given:

$$(\alpha_l, 0) \mapsto (\alpha_{\nu_k(l)}, 0),$$

for $j_{k-1}+1 \leq l \leq j_k - 1$ with $1 \leq k \leq r$, and

$$(\alpha_{j_k}, 0) \mapsto (-\alpha_{j_{k-1}+1} - \cdots - \alpha_{j_{k+1}-1}, 0),$$

for $1 \leq k \leq r-1$, taking $j_0 = 0$.

If $j_r - 1 = n$, then the description is completed by requiring that

$$(\alpha_0, 1) \mapsto (\alpha_{j_1} + \cdots + \alpha_{j_{r-1}}, 1).$$

On the other hand, if $j_r - 1 < n$, then

$$\begin{aligned}
(\alpha_{j_r}, 0) &\mapsto (-\alpha_{j_{r-1}+1} - \cdots - \alpha_{j_r}, 0), \\
(\alpha_k, 0) &\mapsto (-\alpha_k, 0), \ k > j_r, \\
(\alpha_0, 1) &\mapsto (\alpha_{j_1} + \cdots + \alpha_n, 1).
\end{aligned}$$

In lifting, as with the principal case, factors of -1 necessarily arise. When specializing, we need to be sure that the Heisenberg element of minimal positive degree for one of the regular subalgebra factors is fixed by $\tilde{\tau}$. For this to be so, some care is needed when specifying the plus or minus sign on the image of λx_0. If we wish the Heisenberg element of minimal degree of $\mathfrak{a}_{n_k}^{(1)}$ to be fixed, we need to stipulate that

$$\lambda x_0 \mapsto (-1)^{n-n_k} \lambda[x_{j_1}, \ldots, x_m],$$

where

$$m = \begin{cases} j_{r-1} & \text{if } j_r - 1 = n \\ n & \text{if } j_r - 1 < n. \end{cases}$$

This ensures that

$$\lambda[y_{j_k-1}, \ldots, y_{j_{k-1}+1}] \mapsto +\lambda[y_{j_k-1}, \ldots, y_{j_{k-1}+1}].$$

The proof is similar to the case for $\tilde{w}$. This time, we consider the situation when $j_r - 1 = n$, that for $j_r - 1 < n$ proceeding in a similar vein:

$$
\begin{aligned}
\lambda[y_{j_k-1},\dots,y_{j_{k-1}+1}] &= [x_{j_k},x_{j_k+1},\dots,x_{j_{k+1}-1},\dots,x_{j_{r-1}},x_{j_{r-1}+1},\dots,x_{j_r-1},\\
&\quad \lambda x_0,x_1,\dots,x_{j_1-1},\dots,x_{j_{k-2}},x_{j_{k-2}+1},\dots,x_{j_{k-1}-1},x_{j_{k-1}}]\\
&\overset{\tilde{\tau}}{\mapsto} [[y_{j_{k+1}-1},\dots,y_{j_{k-1}+1}],x_{j_{k+1}-1},\dots,x_{j_k+1},\dots,\\
&\quad [y_{j_r-1},\dots,y_{j_{r-2}+1}],x_{j_r-1},\dots,x_{j_{r-1}+1},\\
&\quad (-1)^{n-n_k}\lambda[x_{j_1},\dots,x_{j_{r-1}}],x_{j_1-1},\dots,x_1,\dots,\\
&\quad [y_{j_{k-1}-1},\dots,y_{j_{k-3}+1}],x_{j_{k-1}-1},\dots,x_{j_{k-2}+1},\\
&\quad [y_{j_k-1},\dots,y_{j_{k-2}+1}]]\\
&= (-1)^{n-n_k}[(-1)[y_{j_{k+1}-2},\dots,y_{j_{k-1}+1}],x_{j_{k+1}-2},\dots,x_{j_k+1},\dots,\\
&\quad (-1)[y_{j_r-2},\dots,y_{j_{r-2}+1}],x_{j_r-2},\dots,x_{j_{r-1}+1},\\
&\quad (-1)^{n-n_k}(-1)\lambda[x_{j_1-1},\dots,x_{j_{r-1}}],x_{j_1-2},\dots,x_1,\dots,\\
&\quad (-1)[y_{j_{k-1}-2},\dots,y_{j_{k-3}+1}],x_{j_{k-1}-2},\dots,x_{j_{k-2}+1},\\
&\quad [y_{j_k-1},\dots,y_{j_{k-2}+1}]]\\
&\ \vdots\\
&= (-1)^{n-n_k}[(-1)^{n_{k+1}}[y_{j_k},\dots,y_{j_{k-1}+1}],\dots,\\
&\quad (-1)^{n_r}[y_{j_{r-1}},\dots,y_{j_{r-2}+1}],(-1)^{n_1}\lambda[x_1,\dots,x_{j_{r-1}}],\dots,\\
&\quad (-1)^{n_{k-1}}[y_{j_{k-2}},\dots,y_{j_{k-3}+1}],[y_{j_k-1},\dots,y_{j_{k-2}+1}]]\\
&= (-1)^{n-n_k}[(-1)^{(n_{k+1}+1)+\cdots+(n_r+1)}[y_{j_{r-1}},\dots,y_{j_{k-1}+1}],\\
&\quad (-1)^{n_1}\lambda[x_1,\dots,x_{j_{r-1}}],\\
&\quad (-1)^{(n_2+1)+\cdots+(n_{k-1}+1)+1}[y_{j_k-1},\dots,y_1]]\\
&= (-1)^{n-n_k}\lambda[(-1)^{(n_{k+1}+1)+\cdots(n_r+1)+(n_1+1)}[x_1,\dots,x_{j_{k-1}}],\\
&\quad (-1)^{(n_2+1)+\cdots+(n_{k-1}+1)+1}[y_{j_k-1},\dots,y_1]]\\
&= (-1)^{n-n_k}(-1)^{(n_{k+1}+1)+\cdots(n_r+1)+(n_1+1)+(n_2+1)+\cdots+(n_{k-1}+1)}\\
&\quad \times\ \lambda[y_{j_k-1},\dots,y_{j_{k-1}+1}]\\
&= \lambda[y_{j_k-1},\dots,y_{j_{k-1}+1}]
\end{aligned}
$$

The last line follows from the fact that

$$\sum_{k=1}^{r}(n_k+1)=j_r=n+1,$$

so that

$$n-n_k=(n_{k+1}+1)+\cdots(n_r+1)+(n_1+1)+(n_2+1)+\cdots+(n_{k-1}+1).$$

Use has also been made of the relation

$$j_{k+1}-j_k=n_{k+1}+1.$$

In particular, when dealing with the regular case, all the n_k are the same, so that each Heisenberg element of minimal degree for each factor $\mathfrak{a}_{n_k}^{(1)}$ is preserved by $\tilde{\tau}$. Note also, that for all w, $\tilde{\tau}$ has order 2.

5.3 A Closer Look at the Lift When w Admits a Regular Eigenvector

The desired lifts of w and τ for general $w \in W(\mathfrak{a}_n)$ having been constructed, we now focus our attention on those w admitting a regular eigenvector. According to [14, 15], these fall into two categories for $\mathfrak{a}_n$: the partition $\mathcal{P}=\{p,\ldots,p\}$ of $n+1$, containing r elements, and the partition $\mathcal{P}=\{p,\ldots,p,1\}$ of $n+1$, containing $r+1$ elements.

The first case, $\mathcal{P}=\{p,\ldots,p\}$, is represented by the extended Carter diagram in Figure 5.5, for which the regular subalgebra of $\mathfrak{a}_n$ is $\oplus_{k=1}^{r}\mathfrak{a}_{n_k}$, where $n=rp-1$ and each $n_k=p-1$. Thus, $j_k=kp$, $1\le k\le r-1$, and the subspace, $\mathfrak{h}'$, of fixed points of $w=w_C(\mathfrak{a}_{n_1})\ldots w_C(\mathfrak{a}_{n_r})$ has dimension $r-1$. The order, p, of w is preserved on lifting to $\tilde{w}$. Note that in Figure 5.5, the nodes on the bottom of the diagram represent affine simple roots, so that α_l is to be understood to represent $(\alpha_l,0)$. Moreover, a typical node at the top of the extended Dynkin subdiagram of $\mathfrak{a}_{n_k}$, $n_k=p-1$, represents the root $(-\alpha_{(k-1)p+1}-\cdots-\alpha_{kp-1},1)$. These labels have been omitted in the interests of visual clarity. For future reference, we shall refer to this as Case I of regular type.

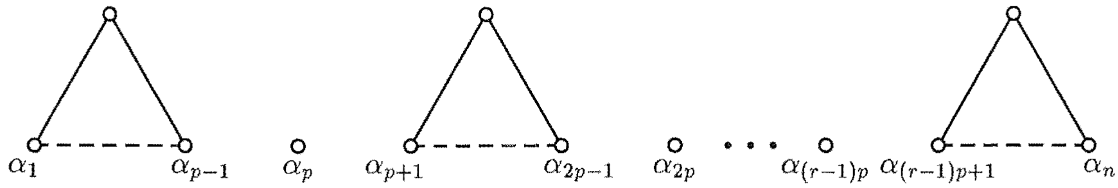

Figure 5.5: Extended Carter diagram for $\mathcal{P} = \{p, \ldots, p\}$

The second case, $\mathcal{P} = \{p, \ldots, p, 1\}$, is represented by the extended Carter diagram in Figure 5.6, for which the regular subalgebra of $\mathfrak{a}_n$ is $\oplus_{k=1}^{r} \mathfrak{a}_{n_k}$, where this time $n = rp$ and again each $n_k = p - 1$. Thus, $j_k = kp$, $1 \leq k \leq r - 1$, and the subspace, $\mathfrak{h}'$, of fixed points of $w = w_C(\mathfrak{a}_{n_1}) \ldots w_C(\mathfrak{a}_{n_r})$ has dimension r. The order, p, of w is preserved on lifting if p is odd, and doubles on lifting if p is even. In other words, the order of $\tilde{w}$ is $\gcd(2, p)p$. The same notational abbreviations apply for Figure 5.6 as before and hereafter this case shall be referred to as Case II of regular type.

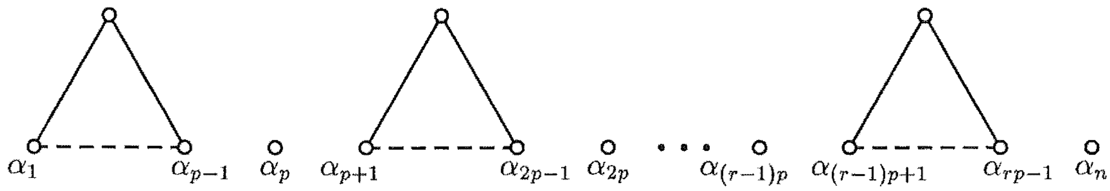

Figure 5.6: Extended Carter diagram for $\mathcal{P} = \{p, \ldots, p, 1\}$

We now describe $\mathcal{H}[w]$ and $\mathcal{H}^{\perp}[w]$ for Cases I and II. Recall that, by construction, $\mathcal{H}[w]$ is the lift to elements of $\mathbf{s}[w]$-homogeneous grade of the image of $\mathfrak{h}$ under ψ, where $\sigma = \psi\tilde{w}\psi^{-1}$ is the diagonal automorphism associated with $\mathbf{s}[w]$. In the representation used throughout this work, ψ was chosen so as to preserve the fixed points of w, so that the Heisenberg subalgebra $\mathcal{H}[w]$ consists of the lifts of the fixed points of $\tilde{w}$ and the principal Heisenberg subalgebras of each of the factors $\mathfrak{a}_{n_k}$ of the regular subalgebra $\oplus_{k=1}^{r}\mathfrak{a}_{n_k}$ of $\mathfrak{a}_n$. The latter elements are easily found using Proposition 5.2.1.

Furthermore, for those w admitting a regular eigenvector, $\psi^{-1}(h)$, $h \in \mathfrak{h}$, $\mathcal{H}^{\perp}[w] =$ Im ad Λ contains $\mathbb{C}[\lambda, \lambda^{-1}]\otimes(\oplus_{k=1}^{r}\mathfrak{h}_{n_k})$ ($n_k = p-1$) as well as the rest of $\mathcal{H}^{\perp}[w_C(\mathfrak{a}_{n_k})]$, again given by Proposition 5.2.1. The description of $\mathcal{H}^{\perp}[w]$ is completed by adding those elments of $\mathfrak{a}_n^{(1)} = \mathcal{H}[w] \oplus \mathcal{H}^{\perp}[w]$ that are linearly independent of $\mathcal{H}[w]$ and the members of $\mathcal{H}^{\perp}[w]$ listed above: namely, the lifts of all the $x_{j_k} = x_{kp}$ and $y_{j_k} = y_{kp}$ as well as all commutators containing these elements. In summary, then:

Proposition 5.3.1 *Given $w \in W(\mathfrak{a}_n)$ admitting a regular eigenvector,*

$$\mathcal{H}[w] = \oplus_{k=1}^{r}\mathcal{H}[w_C(\mathfrak{a}_{n_k})] \oplus \mathbb{C}[\lambda, \lambda^{-1}] \otimes \mathfrak{h}'$$

and

$$\mathcal{H}^{\perp}[w] = \oplus_{k=1}^{r}\mathcal{H}^{\perp}[w_C(\mathfrak{a}_{n_k})] \oplus \mathbb{C}[\lambda, \lambda^{-1}] \otimes \mathfrak{k},$$

where $\mathfrak{k}$ consists of all possible elements of the form

$$[x_{k_1}, \ldots, x_{k_2}], [y_{k_2}, \ldots, y_{k_1}], \quad 1 \le k_1 \le k_2 \le n,$$

where $k_1 \le j_k \le k_2$ for at least one k, $1 \le k \le r$.

REMARK: Note that $\tilde{w}\,|_{\mathcal{H}[w_C(\mathfrak{a}_{n_k})]}=$ id and $\tilde{w}\,|_{\mathfrak{h}'}=$ id, so that $\tilde{w}$ acts as the identity on all of $\mathcal{H}[w]$. Note also that the regular elements of $\mathcal{H}[w]$ of homogeneous grade must necessarily lie in $\oplus_{k=1}^{r}\mathcal{H}[w_C(\mathfrak{a}_{n_k})]$. Any homogeneous Heisenberg element must lie either entirely in $\oplus_{k=1}^{r}\mathcal{H}[w_C(\mathfrak{a}_{n_k})]$ or in $\mathbb{C}[\lambda, \lambda^{-1}]\otimes\mathfrak{h}'$, as the elements of the latter have degree Nm, $m \in \mathbb{Z}$, of which there is none in the former subspace. Moreover, in order to be regular as well, an element must necessarily lie in $\oplus_{k=1}^{r}\mathcal{H}[w_C(\mathfrak{a}_{n_k})]$, due to the fact that regular elements lie in a *unique* CSA [33, 28], and therefore cannot lie in the lift of $\mathfrak{h}$ which, of course, contains $\mathbb{C}[\lambda, \lambda^{-1}] \otimes \mathfrak{h}'$ (recall that any Heisenberg subalgebra is the lift of a CSA). □

We now examine the action of $\tilde{\tau}$ on $\mathcal{H}[w]$, w admitting a regular eigenvector. Now,

$$\tilde{\tau}\,|_{\mathcal{H}[w_C(\mathfrak{a}_{n_k})]}\colon \Lambda' \mapsto (-1)^{\deg_{\mathbf{s}[w_C(\mathfrak{a}_{n_k})]}\Lambda'-1}\Lambda'.$$

When w does *not* double in order on lifting to $\tilde{w}$, as is the case for those w corresponding to the extended Carter diagrams of Figure 5.5 and those of Figure 5.6 with p *odd*, the $\mathbf{s}[w_C(\mathfrak{a}_{n_k})]$-degree of any element of $\mathfrak{a}_{n_k}^{(1)}$ is preserved in the $\mathbf{s}[w]$-gradation on embedding $\mathfrak{a}_{n_k}^{(1)}$ into $\mathfrak{a}_n^{(1)}$. This is a consequence of the fact that the $\mathbf{s}[w]$-degree of a root $\alpha \in \mathfrak{h}_{n_k}^*$ is given by

$$\begin{aligned} N\langle\gamma_{\mathbf{s}[w]}, \alpha\rangle &= N\langle\gamma_{n_1} + \cdots + \gamma_{n_r}, \alpha\rangle \\ &= \langle\gamma_{n_k}, \alpha\rangle \\ &= \deg_{\mathbf{s}[w_C(\mathfrak{a}_{n_k})]}(\alpha), \end{aligned}$$

where $N = p$ is the order of both $\tilde{w}$ and $\tilde{w}_C(\mathfrak{a}_{n_k})$ and $\gamma_{\mathbf{s}[w]} = \gamma_{n_1} + \cdots + \gamma_{n_r}$ where γ_{n_k} is the shift vector for $\mathbf{s}[w_C(\mathfrak{a}_{n_k})]$ and the γ_{n_k} are mutually orthogonal.

For the case of Figure 5.6 when p is *even*, the order doubles on lifting to $\tilde{w}$, so that $N = 2p$, whence it follows that

$$\deg_{\mathbf{s}[w]}(\alpha) = 2\deg_{\mathbf{s}[w_C(\mathfrak{a}_{n_k})]}(\alpha)$$

for $\alpha \in \mathfrak{h}_{n_k}^*$. Given the difficulties previously experienced in developing a specialization theory for this case, we shall restrict our attention only to those w whose order is preserved on lifting. For such w, it is now clear that

$$\tilde{\tau}\mid_{\mathcal{H}[w_C(\mathfrak{a}_{n_k})]}: \Lambda' \mapsto (-1)^{\deg_{\mathbf{s}[w]}\Lambda' - 1}\Lambda'.$$

It remains to examine the action on $\mathbb{C}[\lambda, \lambda^{-1}] \otimes \mathfrak{h}'$, which constitutes the rest of $\mathcal{H}[w]$. However, it was earlier noted in the discussion of $\tilde{\tau}$ for the principal case, that

$$\tilde{\tau}(\lambda^m h_j) = (-1)^{\deg(\lambda^m h_j)}\lambda^m h_{\nu(j)},$$

where $h_j \in \mathfrak{h}$. Thus, for the general regular case where the order is preserved on lifting, if $h_j \in \mathfrak{h}_{n_k}$,

$$\tilde{\tau}(\lambda^m h_j) = (-1)^{\deg_{\mathbf{s}[w]}(\lambda^m h_j)}\lambda^m h_{\nu_k(j)}.$$

When $m = 0$, $\deg_{\mathbf{s}[w]}(h_j) = 0$, so that $\tilde{\tau}(h_j) = h_{\nu_k(j)}$. On the other hand,

$$\tau\mid_{\mathfrak{h}'} = -\mathrm{id}$$

by Theorem 5.1.1. Thus, on $\mathfrak{h}$, $\tilde{\tau}$ has the action

$$h \mapsto \begin{cases} (-1)^{\deg_{s[w]} h}\tau(h) & \text{if } h \in \mathfrak{h}_{n_k}, \\ (-1)^{\deg_{s[w]} h-1} h & \text{if } h \in \mathfrak{h}'. \end{cases}$$

Thus, in order to remain consistent, on lifting

$$\lambda^m h \mapsto \begin{cases} (-1)^{\deg_{s[w]} \lambda^m h}\lambda^m\tau(h) & \text{if } h \in \mathfrak{h}_{n_k}, \\ (-1)^{\deg_{s[w]} \lambda^m h-1} \lambda^m h & \text{if } h \in \mathfrak{h}'. \end{cases}$$

As $\mathbb{C}[\lambda, \lambda^{-1}]$ comprises the rest of $\mathcal{H}[w]$, we have established:

Proposition 5.3.2 *Given $w \in W(\mathfrak{a}_n)$ admitting a regular eigenvector and whose order is preserved on lifting,*

$$\tilde{\tau}(\Lambda') = (-1)^{\deg_{s[w]} \Lambda' - 1}\Lambda'$$

for all (homogeneously graded) $\Lambda' \in \mathcal{H}[w]$.

REMARK: A similar result applies when w doubles order on lifting, namely

$$\tilde{\tau}(\Lambda') = (-1)^{\frac{1}{2}\deg_{s[w]} \Lambda' - 1}\Lambda'$$

for all $\Lambda' \in \mathcal{H}[w]$. Recall that $\mathcal{H}[w]$ consists of elements of even degree only in this case. □

Corollary 5.3.3 *The vector space direct sum decomposition*

$$\mathfrak{a}_n^{(1)} = \mathcal{H}[w] \oplus \mathcal{H}^{\perp}[w]$$

is preserved by the action of $\tilde{\tau}$.

PROOF: By Proposition 4.2.2, there exists a regular Heisenberg element, Λ, of minimal degree (*i.e.* 1 if the order is preserved and 2 if the order doubles). Thus, $\mathcal{H}[w] = \text{Ker ad } \Lambda$ and $\mathcal{H}^{\perp}[w] = \text{Im ad } \Lambda$. Proposition 5.3.2 and the Remark following it show that $\tilde{\tau}(\Lambda) = \Lambda$. Therefore, given $\Lambda' \in \text{Ker ad } \Lambda$,

$$[\Lambda, \tilde{\tau}(\Lambda')] = \tilde{\tau}[\Lambda, \Lambda'] = 0,$$

so that $\tilde{\tau}(\Lambda') \in \mathrm{Ker\ ad}\ \Lambda$, and if $X = [\Lambda, Y] \in \mathrm{Im\ ad}\ \Lambda$, then

$$\tilde{\tau}(X) = [\Lambda, \tilde{\tau}(Y)] \in \mathrm{Im\ ad}\ \Lambda.$$

Thus, $\tilde{\tau}$ preserves the direct sum decomposition of $\mathfrak{a}_n^{(1)}$ as claimed. □

We now show that $\tilde{\tau}$ preserves the $\mathrm{s}[w]$-gradation in the regular case. First, we ascertain $\mathrm{s}[w]$ explicitly.

Theorem 5.3.4 *Let $w \in W(\mathfrak{a}_n)$ admit a regular eigenvector. Then either (i) w corresponds to $\mathcal{P} = \{p, \ldots, p\}$, $n = rp - 1$, whence*

$$\begin{aligned}\mathrm{s}[w] &= (s_0, s_1, \ldots, s_{p-1}, s_p, \ldots, s_{(r-2)p+1}, \ldots, s_{(r-1)p-1}, s_{(r-1)p}, s_{(r-1)p+1}, \ldots, s_n) \\ &= (1, 1, \ldots, 1, -(p-1), \ldots, 1, \ldots, 1, -(p-1), 1, \ldots, 1);\end{aligned}$$

or (ii) w corresponds to $\mathcal{P} = \{p, \ldots, p, 1\}$, $n = rp$, with p odd, whence

$$\begin{aligned}\mathrm{s}[w] &= (s_0, s_1, \ldots, s_{p-1}, s_p, \ldots, s_{(r-2)p+1}, \ldots, s_{(r-1)p-1}, s_{(r-1)p}, \\ &\qquad s_{(r-1)p+1}, \ldots, s_{rp-1}, s_n) \\ &= \left(\tfrac{1}{2}(p+1), 1, \ldots, 1, -(p-1), \ldots, 1, \ldots, 1, -(p-1), 1, \ldots, 1, -\tfrac{1}{2}(p-1)\right);\end{aligned}$$

or (iii) w corresponds to $\mathcal{P} = \{p, \ldots, p, 1\}$, $n = rp$, with p even, whence

$$\begin{aligned}\mathrm{s}[w] &= (s_0, s_1, \ldots, s_{p-1}, s_p, \ldots, s_{(r-2)p+1}, \ldots, s_{(r-1)p-1}, s_{(r-1)p}, \\ &\qquad s_{(r-1)p+1}, \ldots, s_{rp-1}, s_n) \\ &= (p+1, 2, \ldots, 2, -2(p-1), \ldots, 2, \ldots, 2, -2(p-1), 2, \ldots, 2, -(p-1)).\end{aligned}$$

PROOF: First recall the relation of the vector $\mathrm{s}[w]$ defining the gradation associated to w to the corresponding shift vector, namely

$$\gamma_{\mathrm{s}[w]} = \frac{1}{N} \sum_{j=1}^{n} s_j \lambda_j,$$

where the λ_j are the fundamental dominant weights, N is the order of $\tilde{w}$ and $s_0 = N - \sum_{j=1}^{n} s_j$. Recall also that

$$\gamma_{\mathrm{s}[w]} = \gamma_{n_1} + \cdots + \gamma_{n_r},$$

where γ_{n_k} is the principal shift vector for $w_C(\mathfrak{a}_{n_k})$ for $w = w_C(\mathfrak{a}_{n_1}) \dots w_C(\mathfrak{a}_{n_r})$.

We have previously seen that

$$(s_1, \dots, s_n) = N \cdot \mathbf{c}A,$$

where A is the Cartan matrix of $\mathfrak{a}_n$ and

$$\gamma_{\mathbf{s}[w]} = \sum_{k=1}^{n} c_k \alpha_k.$$

So, it is necessary to first determine the coefficients c_k. Now,

$$\begin{aligned}
\gamma_{n_k} &= \frac{1}{p}\rho_{n_k} = \frac{1}{2p} \sum_{\alpha \in \Delta^+_{n_k}} \alpha \\
&= \frac{1}{2p} \sum_{j=(k-1)p+1}^{kp-1} (j-(k-1)p)(kp-j)\alpha_j.
\end{aligned}$$

The last line follows from the fact that in $\mathfrak{a}_m$,

$$\sum_{\alpha \in \Delta^+} \alpha = \sum_{j=1}^{m} j(p-j)\alpha_j.$$

The coefficient of α_j is readily determined by considering the number of elements B_{kl} in a $(m+1) \times (m+1)$ matrix B satisfying $k \geq j$ and $l \leq j+1$. This may be visualised as the submatrix of B with lower left hand corner $b_{j,j+1}$ which corresponds to the x_j position, x_j being in the root space of the simple root α_j. Each such B_{kl} then corresponds to the root space of a positive root containing x_j in its commutator.

Thus,

$$\begin{aligned}
\gamma_{\mathbf{s}[w]} &= \sum_{k=1}^{r} \gamma_{n_k} \\
&= \frac{1}{2p} \sum_{k=1}^{r} \sum_{j=(k-1)p+1}^{kp-1} (j-(k-1)p)(kp-j)\alpha_j \\
&= \frac{1}{2p} \sum_{j=1}^{n} \bar{j}(p-\bar{j})\alpha_j,
\end{aligned}$$

where $\bar{j} \equiv j \bmod p$. Hence,

$$c_j = \frac{1}{2p}\bar{j}(p-\bar{j}), \quad 1 \leq j \leq n.$$

Thus, since the Cartan matrix $A = (a_{jk})$ for $\mathfrak{a}_n$ is given by

$$a_{jk} = 2\delta_{kj} - \delta_{k,j+1} - \delta_{k,j-1},$$

then

$$\begin{aligned} s_k &= \sum_{j=1}^{n} pc_j a_{jk} \\ &= \frac{1}{2}\sum_{j=1}^{n} \bar{j}(p-\bar{j})(2\delta_{kj} - \delta_{k,j+1} - \delta_{k,j-1}). \end{aligned}$$

If $k-1, k, k+1$ are *not* multiples of p, then

$$\begin{aligned} s_k &= \frac{1}{2}\left[(\bar{k}-1)(p-(\bar{k}-1))(-1) + \bar{k}(p-\bar{k})\cdot 2 + (\bar{k}+1)(p-\bar{k}-1)(-1)\right] \\ &= \frac{1}{2}\left[2\bar{k}(p-\bar{k}) - (\bar{k}-1)(p-(\bar{k}-1)) - (\bar{k}+1)(p-(\bar{k}+1))\right] \\ &= \frac{1}{2}\left[2\bar{k}p - (\bar{k}-1)p - (\bar{k}+1)p - 2\bar{k}^2 + (\bar{k}-1)^2 + (\bar{k}+1)^2\right] \\ &= 1. \end{aligned}$$

If $k-1 = lp$ for some l, so that $k \equiv 1 \bmod p$ then

$$s_k = \frac{1}{2p}[1(p-1)\cdot 2 + 2(p-2)(-1)] = 1.$$

Similarly, $s_k = 1$ when $k+1$ is a multiple of p. There remains the case when $k \equiv 0 \bmod p$, whence

$$s_k = \frac{1}{2}[(p-1)(p-(p-1))(-1) + 1(p-1)(-1)] = -(p-1).$$

Thus,

$$s_k = \begin{cases} 1 & \text{if } k \not\equiv 0 \bmod p, \\ -(p-1) & \text{if } k \equiv 0 \bmod p. \end{cases}$$

It follows that $s_0 = p - \sum_{k=1}^{n} s_k = 1$, thus establishing (i).

Much the same argument applies for (ii), the only difference occurring for s_n, as now $n = rp \equiv 0 \bmod p$:

$$
\begin{aligned}
s_n &= \frac{1}{2}\sum_{j=1}^{n}\bar{j}(p-\bar{j})(2\delta_{nj}-\delta_{n,j+1}-\delta_{n,j-1}) \\
&= \frac{1}{2}(\overline{n-1}(p-\overline{n-1})(-1)+\bar{n}(p-\bar{n})\cdot 2) \\
&= \frac{1}{2}((p-1)(p-(p-1))(-1)+0) \\
&= -\frac{1}{2}(p-1).
\end{aligned}
$$

Therefore, $s_0 = p - \sum_{k=1}^{n} s_k = \frac{1}{2}(p+1)$, thus proving (ii). Case (iii) follows immediately on observing that now $N = 2p$, and $\mathbf{c}$ remains the same as in (ii), so that

$$(s_1, \ldots, s_n) = N \cdot \mathbf{c}A = 2p \cdot \mathbf{c}A,$$

giving twice the $\mathbf{s}[w]$-vector in (ii). $\square$

Corollary 5.3.5 *Let $w \in W(\mathfrak{a}_n)$ admit a regular eigenvector. Then the lift $\tilde{\tau}$ of τ defined in Theorem 5.1.1 preserves the* $\mathbf{s}[w]$*-gradation.*

PROOF: It suffices to check that the gradation is preserved on the standard set of generators $\{\lambda x_0, x_1, \ldots, x_n\}$. This is the same as grade preservation of the associated simple roots under τ. In case (i), Theorem 5.1.1 and the construction of the lift give:

$$
\begin{aligned}
(\alpha_j, 0) &\mapsto (\alpha_{\nu_k(j)}, 0) \quad j \in I_k = \{(k-1)p+1, \ldots, kp-1\} \\
(\alpha_{kp}, 0) &\mapsto (-\alpha_{(k-1)p+1} - \cdots - \alpha_{(k+1)p-1}, 0) \quad 1 \le k \le r-1.
\end{aligned}
$$

Both the image amd preimage in the first line have $\mathbf{s}[w]$-degree 1, as given by Theorem 5.3.4, while $\deg_{\mathbf{s}[w]}(\alpha_{kp}, 0) = -(p-1)$. Now,

$$
\begin{aligned}
&\deg_{\mathbf{s}[w]}(-\alpha_{(k-1)p+1} - \cdots - \alpha_{(k+1)p-1}, 0) \\
= \;&\deg_{\mathbf{s}[w]}((-\alpha_{(k-1)p+1} - \cdots - \alpha_{kp-1}, 0) + (-\alpha_{kp}, 0) + (\alpha_{kp+1} - \cdots - \alpha_{(k+1)p-1}, 0)) \\
= \;&(-1 - \cdots - 1) + (p-1) + (-1 - \cdots - 1) \\
= \;&-(p-1).
\end{aligned}
$$

Moreover, $\deg_{s[w]}(\alpha_0, 1) = 1$ and

$$\begin{aligned}
(\alpha_0, 1) &\mapsto (\alpha_p + \cdots + \alpha_{(r-1)p}, 1) \\
&= (\alpha_p, 1) + (\alpha_{p+1} + \cdots + \alpha_{2p-1}, 0) + (\alpha_{2p}, 0) + \cdots \\
&\quad +(\alpha_{(r-2)p+1} + \cdots + \alpha_{(r-1)p-1}, 0) + (\alpha_{(r-1)p}, 0),
\end{aligned}$$

the right hand side of which has $s[w]$-degree

$$(p - (p-1)) + (p-1) - (p-1) + \cdots + (p-1) - (p-1) = 1.$$

A similar argument holds for cases (ii) and (iii), which are identical apart from the factor of 2 introduced by the doubling of order. The only real differences here are that now

$$(\alpha_n, 0) \mapsto (-\alpha_{(r-1)p+1} - \cdots - \alpha_{n-1} - \alpha_n, 0),$$

the left hand side of which has $s[w]$-degree $-\frac{1}{2}(p-1)$ in case (ii) (respectively $-(p-1)$ in case (iii)) and the right hand side degree

$$-\left(1 + \cdots + 1 - \frac{1}{2}(p-1)\right) = -\frac{1}{2}(p-1)$$

(respectively $-(p-1)$). Finally,

$$(\alpha_0, 1) \mapsto (\alpha_p + \cdots + \alpha_n, 1),$$

the left hand side having degree $\frac{1}{2}(p+1)$ (respectively $p+1$) and the right hand side degree

$$p - (p-1) + (p-1) - (p-1) + \cdots + (p-1) - \frac{1}{2}(p-1) = 1 + \frac{1}{2}(p-1) = \frac{1}{2}(p+1)$$

(respectively $p+1$). □

For Case I of regular type, we also have:

Corollary 5.3.6 *Let $w \in W(\mathfrak{a}_n)$ admitting a regular eigenvector be of Case I type, so that w corresponds to $\mathcal{P} = \{p, \ldots, p\}$, $n = rp - 1$. Then the lift $\tilde{w}$, as defined in the previous section, preserves the $s[w]$-gradation.*

PROOF: Recall that the action of the lift on the simple roots of $\mathfrak{a}_n^{(1)}$ is:

$$\begin{aligned}
(\alpha_{(k-1)p+1}, 0) &\mapsto (\alpha_{(k-1)p+2}, 0), \\
(\alpha_{(k-1)p+2}, 0) &\mapsto (\alpha_{(k-1)p+3}, 0), \\
&\vdots \\
(\alpha_{kp-2}, 0) &\mapsto (\alpha_{kp-1}, 0), \\
(\alpha_{kp-1}, 0) &\mapsto (-\alpha_{(k-1)p+1} - \cdots - \alpha_{kp-1}, 1),
\end{aligned}$$

for $1 \leq k \leq r$;

$$(\alpha_{kp}, 0) \mapsto (\alpha_{(k-1)p+1} + \cdots + \alpha_{kp+1}, -1),$$

for $1 \leq k \leq r-1$, taking $j_0 = 0$; and

$$(\alpha_0, 1) \mapsto (-\alpha_2 - \cdots - \alpha_{(r-1)p}, 0).$$

Since, by Theorem 5.3.4,

$$\deg_{\mathbf{s}[w]}(\alpha_{(k-1)p+1}, 0) = \cdots = \deg_{\mathbf{s}[w]}(\alpha_{kp-1}, 0) = 1$$

and

$$\deg_{\mathbf{s}[w]}(-\alpha_{(k-1)p+1} - \cdots - \alpha_{kp-1}, 1) = p - (p-1) = 1,$$

it follows that the gradation is preserved for all $(\alpha_j, 0)$ where $j \in I_k$, $1 \leq k \leq r-1$. Now $\deg_{\mathbf{s}[w]}(\alpha_{kp}, 0) = -(p-1)$ while

$$\deg_{\mathbf{s}[w]}(\alpha_{(k-1)p+1} + \cdots + \alpha_{kp+1}, -1) = -p + 1 + \cdots + 1 - (p-1) + 1 = -(p-1).$$

Also $\deg_{\mathbf{s}[w]}(\alpha_0, 1) = 1$ while

$$\begin{aligned}
\deg_{\mathbf{s}[w]}(-\alpha_2 - \cdots - \alpha_{(r-1)p}, 0) &= -((p-2) - (p-1) + (p-1) - (p-1) \\
&\quad + \cdots + (p-1) - (p-1)) \\
&= 1.
\end{aligned}$$

Thus, the gradation is preserved under the action on the simple roots, as claimed.□

REMARK: For w admitting regular eigenvectors of Case II type, the gradation is *not* preserved under $\tilde{w}$. All the simple roots *except* $(\alpha_n, 0)$ and $(\alpha_0, 1)$ are mapped to roots of the same grade. However,

$$(\alpha_n, 0) \mapsto (\alpha_{(r-1)p+1} + \cdots + \alpha_n, -1),$$

for which the left hand side has grade $-\frac{1}{2}(p-1)$ if the order is preserved on lifting (respectively, $-(p-1)$ if the order doubles), while the right hand side has grade

$$1 + \cdots + 1 - \frac{1}{2}(p-1) = \frac{1}{2}(p-1)$$

(respectively, $p-1$). Furthermore,

$$(\alpha_0, 1) \mapsto (-\alpha_2 - \cdots - \alpha_n, 1),$$

for which the left hand side has grade $\frac{1}{2}(p+1)$ (respectively $p+1$), while the right hand side has grade

$$-((p-2)-(p-1)+(p-1)-(p-1)+\cdots+(p-1)-(p-1)+(p-1)-\tfrac{1}{2}(p-1)) = -\tfrac{1}{2}(p-3),$$

(respectively, $-(p-3)$). As in both cases the grades of the image and preimage differ modulo p (respectively $2p$), there is no way of redefining the lift $\tilde{w}$, by adjusting the powers of λ, so that the gradation is preserved. Since the only freedom in lifting the root space automorphism w is in scalar multiples of the images of elements, or the powers of the parameter λ, there appears to be no other way in which such w may be lifted so as to preserve the gradation. Of course, there still remains the possibility that there is another gradation that is preserved under both $\tilde{\tau}$ and $\tilde{w}$ for regular eigenvectors of Case II type. □

Corollary 5.3.7 *Let $w \in W(\mathfrak{a}_n)$ admitting a regular eigenvector be of Case I type, so that w corresponds to $\mathcal{P} = \{p, \ldots, p\}$, $n = rp - 1$. Then the root automorphism*

$$\tau' = \tau w$$

is of order 2 and preserves the $\mathfrak{s}[w]$-gradation. Therefore, the corresponding lift $\tilde{\tau}' = \tilde{\tau}\tilde{w}$ is also of order 2 and is gradation preserving.

PROOF: That τ' and therefore $\tilde{\tau}'$ preserve the gradation is obvious from Corollaries 5.3.5 and 5.3.6.

It is apparent that τ' has order 2 when restricted to the roots α_j, $j \in I_k$, $1 \leq k \leq r-1$, since the effect of τ' on such roots is merely one of the extended Dynkin diagram reflectional symmetries of Figures 5.1 and 5.2 for $\mathfrak{a}_{n_k}^{(1)}$. It remains, then, to verify that τ' has order 2 on the roots $(\alpha_{kp}, 0)$, $1 \leq k \leq r-1$, and on $(\alpha_0, 1)$.

$$\begin{aligned}
\tilde{\tau}'(\alpha_{kp}, 0) &= \tilde{\tau}\tilde{w}(\alpha_{kp}, 0) \\
&= \tilde{\tau}(\alpha_{(k-1)p+1} + \cdots + \alpha_{kp+1}, -1) \\
&= ((\alpha_{(k-1)p+1} + \cdots + \alpha_{kp-1}) - (\alpha_{(k-1)p+1} + \cdots + \alpha_{(k+1)p-1}) \\
&\quad +\alpha_{(k+1)p-1}, -1) \\
&= (-\alpha_{kp} - \cdots - \alpha_{(k+1)p-2}, -1).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\tilde{\tau}'(-\alpha_{kp} - \cdots - \alpha_{(k+1)p-2}, -1) \\
= \ &\tilde{\tau}(-(\alpha_{(k-1)p+1} + \cdots + \alpha_{kp+1}) - (\alpha_{kp+2} + \cdots + \alpha_{(k+1)p-1}), 0) \\
= \ &\tilde{\tau}(-\alpha_{(k-1)p+1} - \cdots - \alpha_{(k+1)p-1}, 0) \\
= \ &\tilde{\tau}^2(\alpha_{kp}, 0) \\
= \ &(\alpha_{kp}, 0).
\end{aligned}$$

Also,

$$\begin{aligned}
\tilde{\tau}'(\alpha_0, 1) &= \tilde{\tau}\tilde{w}(\alpha_0, 1) \\
&= \tilde{\tau}(-\alpha_2 - \cdots - \alpha_{(r-1)p}, 0) \\
&= \tilde{\tau}(\alpha_{(r-1)p+1} + \cdots + \alpha_n + \alpha_0 + \alpha_1, 0) \\
&= ((\alpha_{(r-1)p+1} + \cdots + \alpha_n) + (\alpha_p + \cdots + \alpha_{(r-1)p} + \alpha_{p-1}), 0) \\
&= (\alpha_{p-1} + \cdots + \alpha_n, 0).
\end{aligned}$$

Hence,

$$\begin{aligned}
& \tilde{\tau}'(\alpha_{p-1} + \cdots + \alpha_n, 0) \\
= \; & \tilde{\tau}'(-(\alpha_0 + \alpha_1 + \cdots + \alpha_{p-2}), 0) \\
= \; & \tilde{\tau}((\alpha_2 + \cdots + \alpha_{(r-1)p}) - (\alpha_2 + \cdots + \alpha_{p-1}), 0) \\
= \; & \tilde{\tau}(\alpha_p + \cdots + \alpha_{(r-1)p}, 0) \\
= \; & \tilde{\tau}^2(\alpha_0, 1) \\
= \; & (\alpha_0, 1).
\end{aligned}$$

This establishes that $\tilde{\tau}'$ is indeed a root space involution as claimed. That the lift to $\mathrm{Aut}(\mathfrak{a}_n^{(1)})$ also has order 2 is due to the fact that, as with $\tilde{\tau}$, the only scalar factors introduced on lifting to $\mathrm{Aut}(\mathfrak{a}_n^{(1)})$ are powers of -1. □

Corollary 5.3.7 will be useful in defining an alternative specialization $\bar{q} = \tilde{\tau}'(q)$ for those w of Case I type, when p is *even*, as then the fixed point subalgebras of $\tilde{\tau}$ and $\tilde{\tau}'$ are not isomorphic, thus the two different automorphisms give rise to two nontrivially different specialized hierarchies of evolutionary systems. Those for the principal case were studied by Guil [25]. Running through the conjugacy classes of $\mathfrak{a}_n$ in increasing order of n, the first non-principal case that would admit such a gradation preserving $\tilde{\tau}'$, with fixed point subalgebra not isomorphic to that of $\tilde{\tau}$, is the conjugacy class of $W(\mathfrak{a}_7)$ corresponding to the partition $\mathcal{P} = \{4, 4\}$ of 8, for which a representative is

$$w = r_{\alpha_1} r_{\alpha_2} r_{\alpha_3} r_{\alpha_5} r_{\alpha_6} r_{\alpha_7}.$$

Chapter 6

Specializations

In this chapter, it is shown that the automorphism $\tilde{\tau}$ may be used to construct a specific specialization of the so-called generalized modified KdV hierarchies. The algebraic essentials and an analysis of the specialization equations are given in §6.1. Some specific examples are presented in §6.2, including a redevelopment of Guil's examples from this generalized viewpoint. Finally, a few concluding remarks are made in §6.3 of the strengths and weaknesses of this theory, with an eye to further possible developments.

6.1 Specializations with the Automorphism $\tilde{\tau}$

It has been implied in the preceding algebraic analysis and the section on gauge symmetries that $\bar{q} = \tilde{\tau}(q)$ gives a specific specialization of our zero-curvature system. This introduces a relation between $\tilde{\tau}(q)$ and q that decreases the dimension of the space that q is constrained to lie in, so reducing the number of independent functions that occur in the resulting evolutionary system. A satisfactory specialization theory has been developed for $w \in W(\mathfrak{a}_n)$ admitting a regular eigenvector and preserving order on being lifted to $\mathrm{Aut}(\mathfrak{a}_n^{(1)})$. To recapitulate, let

$$\mathcal{L}_q = D_x + q + \Lambda,$$

where $\Lambda \in \mathcal{H}_1[w]$ is a regular Heisenberg element of $\mathsf{s}[w]$-degree 1, and $q \in C^\infty(\mathbb{R}, Q)$, where[1] $Q = \mathcal{H}_0^\perp[w]$. Suppose $\Lambda' \in \mathcal{H}_k[w]$ is any Heisenberg element (not necessarily regular) of degree k, and

$$a(q) = \sum_{j=0}^{k-1} a_j(q),$$

where each $a_j(q) \in C^\infty(\mathbb{R}, (\mathfrak{a}_n^{(1)})_j(\mathsf{s}[w]))$ and $(\mathfrak{a}_n^{(1)})_j(\mathsf{s}[w])$ is the subspace of $\mathfrak{a}_n^{(1)}$ of $\mathsf{s}[w]$-degree j. The associated zero-curvature system of evolutionary equations is

$$[\mathcal{L}_q, D_t + a(q) + \Lambda'] = 0. \tag{6.1}$$

Proposition 4.3.1 ensures that the gauge transformation

$$\mathcal{L}_{\bar{q}} = K\mathcal{L}_q K^{-1}, \tag{6.2}$$

where

$$K = I + \sum_{j>0} X_j(q, \bar{q}),$$

with $X_j \in C^\infty(\mathbb{R}, (\mathfrak{a}_n^{(1)})_{-j}(\mathsf{s}[w]))$ has a unique solution consistent with the integrability conditions given by (6.1). Moreover, Proposition 4.3.2 guarantees that the finite subsystem given by $X_j = 0$ for some $j > 0$ (which forces all $X_m = 0$ for $m \geq j$) is also consistent with (6.1). In particular, imposing $X_2 = 0$ in (6.1), and writing $X = X_1$, gives:

$$\bar{q} - q + [\Lambda, X] = 0 \tag{6.3}$$

$$D_x X + \bar{q}X - Xq = 0 \tag{6.4}$$

This follows from (4.8) and (4.9).

We now show that setting $\bar{q} = \tilde{\tau}(q)$ is consistent with (6.1) provided k, the degree of Λ', is odd. To be more precise, on separating (6.1) into its constituent $\mathsf{s}[w]$-grade parts, we obtain:

[1] Recall that there is no loss of generality in ignoring the $\mathcal{H}_0[w]$-component of q, as shown in [13].

$$[q, \Lambda'] + [\Lambda, a_{k-1}(q)] = 0 \tag{6.5}$$

$$D_x a_{k-1}(q) + [q, a_{k-1}(q)] + [\Lambda, a_{k-2}(q)] = 0 \tag{6.6}$$

$$\vdots$$

$$D_x a_1(q) + [q, a_1(q)] + [\Lambda, a_0(q)] = 0 \tag{6.7}$$

$$-D_t q + D_x a_0(q) + [q, a_0(q)] = 0 \tag{6.8}$$

A consistent zero-curvature system is required when q is replaced by $\bar{q}$ in (6.1). Thus $\bar{q} = \tilde{\tau}(q)$ gives a specialization of (6.1) if replacing q by $\bar{q} = \tilde{\tau}(q)$ and $a_j(q)$ by $\bar{a}_j(\bar{q}) = \bar{a}_j(\tilde{\tau}(q))$ yields a set of equations consistent with the constituent parts of (6.1).

Theorem 6.1.1 *If $\bar{q} = \tilde{\tau}(q)$, and $\Lambda' \in \mathcal{H}[w]$ is of* odd *degree k, then*

$$[\mathcal{L}_{\bar{q}}, D_t + \bar{a}(\bar{q}) + \Lambda'] = \tilde{\tau}[\mathcal{L}_q, D_t + a(q) + \Lambda'],$$

in which case, $\bar{q} = \tilde{\tau}(q)$ gives a specialization of (6.1).

PROOF: We need to show that applying $\tilde{\tau}$ to Equations (6.5)–(6.8) is the same as replacing q by $\bar{q} = \tilde{\tau}(q)$ and $a_j(q)$ by $\bar{a}_j(\bar{q}) = \bar{a}_j(\tilde{\tau}(q))$ in the same equations. Now, if k is odd, Proposition 5.3.2 tells us that $\tilde{\tau}(\Lambda') = \Lambda'$. Since the $\mathrm{s}[w]$-degree of Λ is 1, $\tilde{\tau}$ fixes Λ as well. Thus, the effect of $\tilde{\tau}$ on Equation (6.5) is:

$$[\bar{q}, \Lambda'] + [\Lambda, \tilde{\tau}(a_{k-1}(q))] = 0.$$

Comparing this with the corresponding equation from the zero-curvature system in $\bar{q}$ and $\bar{a}(\bar{q})$, namely,

$$[\bar{q}, \Lambda'] + [\Lambda, \bar{a}_{k-1}(\bar{q})] = 0,$$

it follows that

$$[\Lambda, \tilde{\tau}(a_{k-1}(q)) - \bar{a}_{k-1}(\bar{q})] = 0,$$

so that the $\mathcal{H}^{\perp}[w]$-component of $\tilde{\tau}(a_{k-1}(q)) - \bar{a}_{k-1}(\bar{q})$ is zero. However, Corollary 5.3.3 ensures that $\tilde{\tau}$ preserves the complementary subspaces $\mathcal{H}[w]$ and $\mathcal{H}^{\perp}[w]$ so that

$$\tilde{\tau}(a_{k-1}(q)^{\perp}) = \bar{a}_{k-1}(\bar{q})^{\perp}.$$

Next, as $q \in \mathcal{H}_0^{\perp}[w]$, Corollaries 5.3.3 and 5.3.5 guarantee that $\bar{q} = \tilde{\tau}(q) \in \mathcal{H}_0^{\perp}[w]$ also. Therefore, extracting the $\mathcal{H}[w]$-component after applying $\tilde{\tau}$ to Equation (6.6), we obtain

$$D_x\tilde{\tau}(a_{k-1}(q)^{\parallel}) + [\bar{q}, \bar{a}_{k-1}(\bar{q})^{\perp}]^{\parallel} = 0,$$

having made use of $\tilde{\tau}(a_{k-1}(q)^{\perp}) = \bar{a}_{k-1}(\bar{q})^{\perp}$. The corresponding equation from the zero-curvature system in $\bar{q}$ and $\bar{a}(\bar{q})$ is

$$D_x\bar{a}_{k-1}(\bar{q})^{\parallel} + [\bar{q}, \bar{a}_{k-1}(\bar{q})^{\perp}]^{\parallel} = 0.$$

Thus,

$$D_x(\tilde{\tau}(a_{k-1}(q)) - \bar{a}_{k-1}(\bar{q}))^{\parallel} = 0,$$

and the homogeneity requirements of differential degree then give

$$\tilde{\tau}(a_{k-1}(q)^{\parallel}) = \bar{a}_{k-1}(\bar{q})^{\parallel}.$$

Hence, $\tilde{\tau}(a_{k-1}(q)) = \bar{a}_{k-1}(\bar{q})$.

Recursively administering the same sort of argument to the remaining equations (6.6)-(6.8) by breaking them up into their $\mathcal{H}[w]$ and $\mathcal{H}^{\perp}[w]$ parts, it follows that $\tilde{\tau}(a_j(q)) = \bar{a}_j(\bar{q})$, for $0 \leq j \leq k-1$, so that replacing q by $\bar{q} = \tilde{\tau}(q)$ results in a zero-curvature system consistent with the original one. □

Now that we have an actual candidate for $\bar{q}$, we can analyse the specialization equations (6.3), (6.4) arising from the gauge transformation when $X_2(q,\bar{q}) = 0$. Of course, this is the simplest of the specializations available. In Appendix A, we look at an example of what happens when $X_3(q,\bar{q}) = 0$, so that $K = I + X_1 + X_2$. Before investigating the specialization equations, we prove the following result:

Lemma 6.1.2 *If $X(q,\bar{q})$ is of* $\mathrm{s}[w]$*-degree* -1*, such that $\tilde{\tau}(X^{\perp}) = -X^{\perp}$, then*

$$(\tilde{\tau}(q)X^{\perp} - X^{\perp}q)^{\parallel} = (-qX^{\perp} + X^{\perp}\tilde{\tau}(q))^{\parallel}.$$

PROOF:

$$(\tilde{\tau}(q)X^{\perp} - X^{\perp}q + qX^{\perp} - X^{\perp}\tilde{\tau}(q))^{\|} = [\tilde{\tau}(q) + q, X^{\perp}]^{\|} \in \mathcal{H}_{-1}[w].$$

By Proposition 5.3.2, $\tilde{\tau}\mid_{\mathcal{H}_{-1}[w]} = \mathrm{id}$, hence,

$$\begin{aligned}
[\tilde{\tau}(q) + q, X^{\perp}]^{\|} &= \tilde{\tau}[\tilde{\tau}(q) + q, X^{\perp}]^{\|} \\
&= [q + \tilde{\tau}(q), \tilde{\tau}(X^{\perp})]^{\|} \\
&= -[\tilde{\tau}(q) + q, X^{\perp}]^{\|},
\end{aligned}$$

so that $[\tilde{\tau}(q) + q, X^{\perp}]^{\|} = 0$, thus proving that

$$(\tilde{\tau}(q)X^{\perp} - X^{\perp}q)^{\|} = (-qX^{\perp} + X^{\perp}\tilde{\tau}(q))^{\|}.$$

□

Theorem 6.1.3 *Let $\bar{q} = \tilde{\tau}(q)$ and $X_2 = 0$ in the gauge transformation (6.2), so that $K = I + X(q, \tilde{\tau}(q))$ where $X \in C^{\infty}(\mathbb{R}, (\mathfrak{a}_n^{(1)})_{-1}(\mathfrak{s}[w]))$. Then the specialization equations (6.3), (6.4) imply that $X \in C^{\infty}(\mathbb{R}, \mathcal{H}_{-1}^{\perp}[w])$ and $\tilde{\tau}(X) = -X$.*

PROOF: Equation (6.3) is now

$$\tilde{\tau}(q) - q + [\Lambda, X] = 0,$$

and is independent of $X^{\|}$. Moreover, as $\tilde{\tau}$ has order 2, the left hand side of the equation is an eigenvector of $\tilde{\tau}$ associated with the eigenvalue -1. Therefore,

$$\tilde{\tau}[\Lambda, X^{\perp}] = [\Lambda, \tilde{\tau}(X^{\perp})] = -[\Lambda, X^{\perp}],$$

so that $\tilde{\tau}(X^{\perp}) = -X^{\perp}$, since $\mathcal{H}[w] \cap \mathcal{H}^{\perp}[w] = 0$. Now, the $\mathcal{H}[w]$-component of (6.4) is

$$D_x X^{\|} + (\tilde{\tau}(q)X^{\perp} - X^{\perp}q)^{\|} = 0.$$

Note that only $X^{\perp}$ is required in the second term, as $q, \tilde{\tau}(q)$ take values in $\mathcal{H}^{\perp}[w]$ and $[\mathcal{H}[w], \mathcal{H}^{\perp}[w]] \subset \mathcal{H}^{\perp}[w]$. We claim that

$$Y := (\tilde{\tau}(q)X^{\perp} - X^{\perp}q)^{\|} = 0.$$

Since $\tilde{\tau}(X^{\perp}) = -X^{\perp}$,

$$\begin{aligned} 2Y &= (\tilde{\tau}(q)X^{\perp} - X^{\perp}q)^{\parallel} + (-qX^{\perp} + X^{\perp}\tilde{\tau}(q))^{\parallel} \text{ by Lemma 6.1.2} \\ &= ((\tilde{\tau}(q) - q)X^{\perp} + X^{\perp}(\tilde{\tau}(q) - q))^{\parallel} \\ &= -([\Lambda, X^{\perp}]X^{\perp} + X^{\perp}[\Lambda, X^{\perp}])^{\parallel} \text{ by (6.3)} \\ &= -[\Lambda, (X^{\perp})^2]^{\parallel}. \end{aligned}$$

The matrix representation of $(X^{\perp})^2$ may have nonzero trace, and thus not represent an element of $\mathfrak{a}_n^{(1)}$. However, if trace $(X^{\perp})^2 = c$, then

$$(X^{\perp})^2 = \tfrac{c}{n+1}I + Z,$$

where I is the identity matrix and Z is some trace free matrix representing an element of $\mathfrak{a}_n^{(1)}$. Thus,

$$Y = -\tfrac{1}{2}[\Lambda, Z]^{\parallel} = 0,$$

as $[\Lambda, Z] \in \mathcal{H}^{\perp}[w]$. Consequently, $D_x X^{\parallel} = 0$, and the usual homogeneity of differential degree arguments yield $X^{\parallel} = 0$, so that $X \in C^{\infty}(\mathbb{R}, \mathcal{H}^{\perp}_{-1}[w])$ and $\tilde{\tau}X = -X$, as required. □

In light of Theorem 6.1.3, the specialization equations are now:

$$\tilde{\tau}(q) - q + [\Lambda, X] = 0 \tag{6.9}$$

$$D_x X + \tilde{\tau}(q)X - Xq = 0 \tag{6.10}$$

where $X \in \mathcal{H}^{\perp}_{-1}[w]$ such that $\tilde{\tau}(X) = -X$. Ultimately, this will result in a system which contains a smaller number of independent functions than the unspecialized system, which has dim $(\mathcal{H}^{\perp}_0(q))$ independent functions. In fact, it turns out that the number of independent functions in the specialized system is the dimension of the fixed point subspace of $\tilde{\tau}\ |_{\mathcal{H}^{\perp}_0[w]}$. To this end, consider the direct sum decomposition of $\mathcal{H}^{\perp}_0[w]$ into the fixed point subspace of $\tilde{\tau}$ and an appropriate complement, to be determined in due course. Thus, let

$$q = R + S_0 + S_1,$$

where $R, S_0, S_1 \in C^\infty(\mathbb{R}, \mathcal{H}_0^\perp[w])$ such that

$$\begin{aligned}\tilde{\tau}(R) &= R,\\ \tilde{\tau}(S_0) &= S_0,\\ \tilde{\tau}(S_1) &= -S_1.\end{aligned}$$

S_1 may be expressed in terms of X, as follows from (6.9):

$$S_1 = \tfrac{1}{2}[\Lambda, X] \tag{6.11}$$

Now (6.10) takes the form:

$$D_x X + [R, X] + [S_0, X] - (S_1 X + X S_1) = 0 \tag{6.12}$$

In order that X, S_0 and S_1 all depend on R, so that the resulting evolutionary system has the required number of independent functions, we solve for S_0 such that (6.12) is independent of S_0 and S_1.

Theorem 6.1.4 *Each term of (6.12) takes values in $\mathcal{H}_{-1}^\perp[w]$, and (6.12) reduces to*

$$D_x X + [R, X] = 0$$

on defining

$$S_0 := \tfrac{1}{2}\left((\Lambda X + X\Lambda) - \tfrac{k}{n+1} I\right)^\perp,$$

where

$$k = \mathit{trace}\ \tfrac{1}{2}(\Lambda X + X\Lambda).$$

PROOF: Recall that $\tilde{\tau}\ |_{\mathcal{H}_{-1}[w]} = \mathrm{id}$ and $\tilde{\tau}(X) = -X$, so that

$$[R, X]^{\|} = \tilde{\tau}[R, X]^{\|} = [\tilde{\tau}(R), \tilde{\tau}(X)]^{\|} = -[R, X]^{\|}.$$

Hence, $[R, X]$ takes values in $\mathcal{H}_{-1}^\perp[w]$ and likewise for $[S_0, X]$. Therefore, the same applies for $S_1 X + X S_1$. In order that (6.12) reduces to

$$D_x X + [R, X] = 0,$$

we require that

$$\begin{aligned}
[S_0, X] &= S_1 X + X S_1 \\
&= \tfrac{1}{2}([\Lambda, X]X + X[\Lambda, X]) \text{ by (6.11)} \\
&= \tfrac{1}{2}[\Lambda, X^2] \\
&= \tfrac{1}{2}[\Lambda, Z^\perp],
\end{aligned}$$

where $X^2 = \frac{c}{n+1}I + Z$, Z being of s$[w]$-degree -2, as in the proof of Theorem 6.1.3. Note also that $\tilde{\tau}(Z) = -Z$ by necessity (of course, $\tilde{\tau}(Z^{\|}) = -Z^{\|}$, by Proposition 5.3.2). Now, if

$$S_0 = \tfrac{1}{2}\left((\Lambda X + X\Lambda) - \tfrac{k}{n+1}I\right)^\perp,$$

then

$$\begin{aligned}
[S_0, X] &= \tfrac{1}{2}[\Lambda X + X\Lambda, X] \\
&= \tfrac{1}{2}(\Lambda X^2 - X\Lambda X + X\Lambda X - X^2\Lambda) \\
&= \tfrac{1}{2}[\Lambda, X^2] \\
&= S_1 X + X S_1,
\end{aligned}$$

the last line having been established above. □

Corollary 6.1.5 *The specialized form of q is*

$$q = R + S,$$

where $R, S \in C^\infty(\mathbb{R}, \mathcal{H}_0^\perp[w])$ with

$$\tilde{\tau}(R) = R$$

and

$$S = (\Lambda X)'^{\perp},$$

where $(\Lambda X)'$ denotes the trace free part of the matrix ΛX. Consequently, the specialization equation is just

$$D_x X + [R, X] = 0, \tag{6.13}$$

which may be solved as a first order linear differential system by setting

$$R := r_x,$$

thus expressing the coefficients of X, and therefore S, in terms of those of r. It follows that the evolutionary system so obtained is a zero-curvature system over the (not necessarily simple) Lie algebra of fixed points of $\tilde{\tau}$, and the number of independent functions occurring in the system is just the dimension of the subspace of $\mathcal{H}_0^{\perp}[w]$ given by the fixed points of $\tilde{\tau}$.

PROOF: By Theorem 6.1.4 and (6.11),

$$\begin{aligned} S = S_0 + S_1 &= \tfrac{1}{2}((\Lambda X + X\Lambda)'^{\perp} + [\Lambda, X]) \\ &= \tfrac{1}{2}(\Lambda X + X\Lambda + [\Lambda, X])'^{\perp} \\ &= (\Lambda X)'^{\perp}. \end{aligned}$$

□

6.2 Examples

6.2.1 The Conjugacy Class of the Coxeter Element w_C in $\mathfrak{a}_n$

The extended Carter diagram in this case is given in Figure 6.1 and the effect of $\tilde{\tau}$ on the regular subalgebra indicated by the dotted axis of reflection. This is the most familiar instance of a conjugacy class admitting a regular eigenvector and the order of w_C is preserved on lifting. It is well known that

$$\mathsf{s}[w] = (1, 1, \ldots, 1).$$

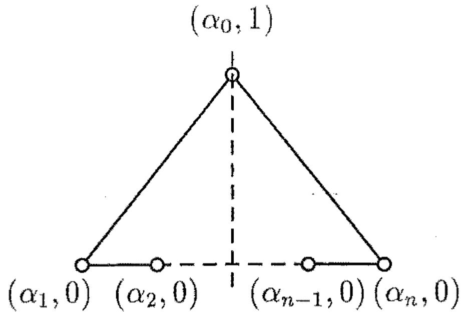

Figure 6.1: Extended Carter diagram for w_C

The principal Heisenberg subalgebra has already been described in Proposition 5.2.2. The typical (regular) principal Heisenberg element of degree 1 is:

$$\Lambda = \lambda x_0 + x_1 + x_2 + \cdots + x_n = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \lambda & & & 0 \end{pmatrix}.$$

From §5.2, the action of $\tilde{w_C}$ on the simple roots of $\mathfrak{a}_n^{(1)}$ is:

$$\begin{aligned} (\alpha_1, 0) &\mapsto (\alpha_2, 0) \\ (\alpha_2, 0) &\mapsto (\alpha_3, 0) \\ &\vdots \\ (\alpha_{n-1}, 0) &\mapsto (\alpha_n, 0) \\ (\alpha_n, 0) &\mapsto (\alpha_0, 1) \\ (\alpha_0, 1) &\mapsto (\alpha_1, 0) \end{aligned}$$

Likewise, the action of $\tilde{\tau}$ is:

$$\begin{aligned} (\alpha_1, 0) &\mapsto (\alpha_n, 0) \\ (\alpha_2, 0) &\mapsto (\alpha_{n-2}, 0) \\ &\vdots \\ (\alpha_k, 0) &\mapsto (\alpha_{n+1-k}, 0) \end{aligned}$$

$$\begin{array}{ccc} & \vdots & \\ (\alpha_n, 0) & \mapsto & (\alpha_1, 0) \\ (\alpha_0, 1) & \mapsto & (\alpha_0, 1) \end{array}$$

In accordance with Proposition 5.3.1,

$$\mathcal{H}_0^{\perp}[w_C] = \mathfrak{h},$$

of which the subspace of fixed points of $\tilde{\tau}$ is spanned by

$$\{h_1 + h_n, h_2 + h_{n-1}, \ldots, h_K + h_{n-K}\},$$

where $K = [(n+1)/2]$ and $[\cdot]$ denotes the integer part. Thus,

$$\begin{aligned} R = r_x &= u_{1_x}(h_1 + h_n) + u_{2_x}(h_2 + h_{n-1}) + \cdots + u_{K_x}(h_K + h_{n-K}) \\ &= \begin{pmatrix} u_{1_x} & & & & \\ & -u_{1_x} + u_{2_x} & & & \\ & & \ddots & & \\ & & & u_{1_x} - u_{2_x} & \\ & & & & -u_{1_x} \end{pmatrix}, \end{aligned}$$

where $u_1, \ldots, u_K$ are functions of x and t. On the other hand, by Proposition 5.2.2, $\mathcal{H}_{-1}^{\perp}[w_C]$ is spanned by

$$\{\lambda^{-1}x_0 - y_1, \ldots, \lambda^{-1}x_0 - y_n\}$$

and the subspace of eigenvectors of $\tilde{\tau}$ with eigenvalue -1 is spanned by

$$\{y_1 - y_n, y_2 - y_{n-1}, \ldots, y_K - y_{n-K}\}.$$

Thus,

$$X = A_1(y_1 - y_n) + A_2(y_2 - y_{n-1}) + \cdots + A_K(y_K - y_{n-K})$$

$$
= \begin{pmatrix} 0 & & & & & \\ A_1 & \ddots & & & & \\ & A_2 & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & -A_2 & 0 & \\ & & & & -A_1 & \end{pmatrix}.
$$

Appealing to Corollary 6.1.5, we obtain:

$$
S = (\Lambda X)'^{\perp} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \lambda & & & 0 \end{pmatrix} \begin{pmatrix} 0 & & & & & \\ A_1 & \ddots & & & & \\ & A_2 & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & -A_2 & 0 & \\ & & & & -A_1 & \end{pmatrix}
$$

$$
= \begin{pmatrix} A_1 & & & & & \\ & A_2 & & & & \\ & & \ddots & & & \\ & & & -A_2 & & \\ & & & & -A_1 & \\ & & & & & 0 \end{pmatrix}
$$

The specialization equation is

$$
D_x X + [r_x, X] = 0.
$$

Now,

$$
r_x = \sum_{k=1}^{K} u_{k_x}(h_k + h_{n+1-k})
$$

and

$$
X = \sum_{l=1}^{K} A_l(y_l - y_{n+1-l}),
$$

so that

$$\begin{aligned}
[r_x, X] &= \sum_{k,l} u_{k_x} A_l [h_k + h_{n+1-k}, y_l - y_{n+1-l}] \\
&= -\sum_{k,l} u_{k_x} A_l (\alpha_l (h_k + h_{n+1-k}) y_l - \alpha_{n+1-l}(h_k + h_{n+1-k}) y_{n+1-l}) \\
&= -\sum_{k,l} u_{k_x} A_l \alpha_l (h_k + h_{n+1-k})(y_l - y_{n+1-l}) \\
&= -\sum_{l=1}^{K} \alpha_l(r_x) A_l (y_l - y_{n+1-l}),
\end{aligned}$$

where use has been made of the fact that

$$[h_i, y_j] = -\alpha_j(h_i) y_j = -a_{ij} y_j,$$

where a_{ij} is the (i,j)-entry of the Cartan matrix of $\mathfrak{a}_n$, namely $2\delta_{ij} - \delta_{i,j-1} - \delta_{i,j+1}$. Thus,

$$\alpha_l(h_k + h_{n+1-k}) = \alpha_{n+1-l}(h_k + h_{n-k}).$$

Thus, the specialization equation yields

$$D_x A_l = \alpha_l(r_x) A_l = (\alpha_l(r))_x A_l,$$

appealing to the the linearity of the typical simple root α_l. Solving leads to

$$A_l = c_l \exp(\alpha_l(r)).$$

Thus,

$$X = \sum_{l=1}^{K} c_l \exp(\alpha_l(r))(y_l - y_{n+1-l}),$$

whence,

$$S = \begin{pmatrix}
c_1 \exp(\alpha_1(r)) & & & & & \\
& c_2 \exp(\alpha_2(r)) & & & & \\
& & \ddots & & & \\
& & & -c_2 \exp(\alpha_2(r)) & & \\
& & & & -c_1 \exp(\alpha_1(r)) & \\
& & & & & 0
\end{pmatrix},$$

and so the specialized form of q,

$$q = r_x + S,$$

is now completely determined. These were the specializations originally studied by Guil in [25]. Most notably, in the case when $n = 3$, the specialization derived from the alternative automorphism $\tilde{\tau}' = \tilde{\tau}\tilde{w}_C$ of Corollary 5.3.7 results in the Calogero-Degasperis equation [8]:

$$p_t = 2p_{xxx} - p_x^3 - 3\left(c_1^2 e^{2p} + c_0^2 e^{-2p}\right) p_x.$$

This may be regarded as a "deformation" of the potential modified Korteweg-de Vries equation, which results from setting the constants c_0, c_1 to zero, giving a deformation of a zero-curvature system over the fixed point subalgebra of $\tilde{\tau}'$, which is $\mathfrak{a}_1^{(1)}$ in this case [25]. Of course, the connection between the modified Korteweg-de Vries equation and $\mathfrak{a}_1$ has long been known.

REMARK: In the principal case, Guil [25] constructs S by showing that it takes values in the fixed point subspace contained in $\mathcal{H}_0^{\perp}[w_C]$ of the automorphism $\tilde{\tau}\tilde{w}_C$. It turns out that the fixed point subspaces of $\tilde{\tau}$ and $\tilde{\tau}\tilde{w}_C$ are complementary in $\mathcal{H}_0^{\perp}[w_C]$. However, there is no immediately obvious way of generalizing this approach for arbitrary conjugacy classes of the Weyl group, as in general, even if $\tilde{\tau}\tilde{w}$ does preserve the $\mathbf{s}[w]$-gradation (see Corollary 5.3.7), the fixed point subspaces of $\tilde{\tau}$ and $\tilde{\tau}\tilde{w}_C$ are *not* complementary, the problem being that, except for the Coxeter case, $\tilde{w}$ always has fixed points lying in $\mathcal{H}_0^{\perp}[w]$, namely the elements $\mathfrak{h}'$, as discussed in §5.1. It would, however, be desirable to develop a generalization without recourse to the matrix representation, in contrast to what has been done here in Corollary 6.1.5, so as to allow the *immediate* possibility of extending the idea to those Kač-Moody algebras apart from $\mathfrak{a}_n^{(1)}$. □

6.2.2 The Conjugacy Class of $r_{\alpha_1} r_{\alpha_2}$ in $\mathfrak{a}_3$

The extended Carter diagram in this case is given in Figure 6.2 and the effect of $\tilde{\tau}$ on the regular subalgebra indicated by the dotted axis of reflection. As an instance

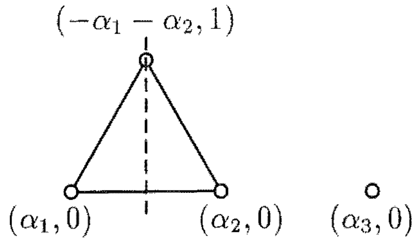

Figure 6.2: Extended Carter diagram for $w = r_{\alpha_1} r_{\alpha_2}$

of Case II of regular type with $p = 3$ being odd, the order of w is preserved on lifting. From Theorem 5.3.4, we know that

$$\mathrm{s}[w] = (2,1,1,-1),$$

in accordance with Example 3.4.2. Moreover, Theorem 5.3.4, as once again verified by Example 3.4.2, shows that $\mathcal{H}[w]$ is generated by:

$$\begin{aligned} \lambda^n(h_1+2h_2+3h_3) &\quad \text{with degree} \quad 3n, \\ \lambda^n(\lambda[y_2,y_1]+x_1+x_2) &\quad \text{with degree} \quad 3n+1, \\ \lambda^n(\lambda y_1+\lambda y_2+[x_1,x_2]) &\quad \text{with degree} \quad 3n+2. \end{aligned}$$

The last two of these give the regular elements. Thus, the typical regular Heisenberg element of degree 1 is:

$$\Lambda = \lambda[y_2,y_1]+x_1+x_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From §5.2, the action of $\tilde{w}$ on the simple roots of $\mathfrak{a}_3^{(1)}$ is:

$$\begin{aligned} (\alpha_1,0) &\mapsto (\alpha_2,0) \\ (\alpha_2,0) &\mapsto (-\alpha_1-\alpha_2,1) \\ (\alpha_3,0) &\mapsto (\alpha_1+\alpha_2+\alpha_3,-1) \\ (\alpha_0,1) &\mapsto (-\alpha_2-\alpha_3,1) \end{aligned}$$

Likewise, the action of $\tilde{\tau}$ is:

$$\begin{aligned}
(\alpha_1, 0) &\mapsto (\alpha_2, 0) \\
(\alpha_2, 0) &\mapsto (\alpha_1, 0) \\
(\alpha_3, 0) &\mapsto (-\alpha_1 - \alpha_2 - \alpha_3, 0) \\
(\alpha_0, 1) &\mapsto (\alpha_3, 1)
\end{aligned}$$

In accordance with Proposition 5.3.1, $\mathcal{H}_0^{\perp}[w]$ is spanned by

$$\{h_1, h_2, [x_2, x_3], [y_3, y_2]\}.$$

Now,

$$\tilde{\tau}[x_2, x_3] = [\tilde{\tau}(x_2), \tilde{\tau}(x_3)] = [x_1, x_0] = -[y_3, y_2],$$

so that the subspace of $\mathcal{H}_0^{\perp}[w]$ of fixed points of $\tilde{\tau}$ is spanned by

$$\{h_1 + h_2, [x_2, x_3] - [y_3, y_2]\}.$$

Hence,

$$R = r_x = u_x(h_1 + h_2) + v_x([x_2, x_3] - [y_3, y_2]) = \begin{pmatrix} u_x & 0 & 0 & 0 \\ 0 & 0 & 0 & v_x \\ 0 & 0 & -u_x & 0 \\ 0 & -v_x & 0 & 0 \end{pmatrix},$$

where u, v are functions of x and t. On the other hand, by Proposition 5.3.1, $\mathcal{H}_{-1}^{\perp}[w]$ is spanned by

$$\{\lambda^{-1}[x_1, x_2] - y_1, \lambda^{-1}[x_1, x_2] - y_2, x_3, x_0\}$$

and the subspace of eigenvectors of $\tilde{\tau}$ with eigenvalue -1 is spanned by

$$\{y_1 - y_2, x_0 - x_3\}.$$

Thus,

$$X = A(y_1 - y_2) + B(x_0 - x_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & -A & 0 & -B \\ B & 0 & 0 & 0 \end{pmatrix}.$$

Now we are in a position to solve the specialization equation (6.13) of Corollary 6.1.5. First, we calculate $S = (\Lambda X)'^{\perp}$:

$$\begin{aligned}\Lambda X &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & -A & 0 & -B \\ B & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & -A & 0 & -B \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

As the last matrix is trace free and has no $\mathcal{H}_0[w]$-component,

$$S = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & -A & 0 & -B \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The specialization equation is

$$D_x X + [r_x, X] = 0,$$

of which the left hand side is

$$\begin{aligned}&\begin{pmatrix} 0 & 0 & 0 & 0 \\ A_x & 0 & 0 & 0 \\ 0 & -A_x & 0 & -B_x \\ B_x & 0 & 0 & 0 \end{pmatrix} \\ &+ \left[\begin{pmatrix} u_x & 0 & 0 & 0 \\ 0 & 0 & 0 & v_x \\ 0 & 0 & -u_x & 0 \\ 0 & -v_x & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & -A & 0 & -B \\ B & 0 & 0 & 0 \end{pmatrix} \right]\end{aligned}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_x & 0 & 0 & 0 \\ 0 & -A_x & 0 & -B_x \\ B_x & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ v_xB & 0 & 0 & 0 \\ 0 & u_xA & 0 & u_xB \\ -v_xA & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ u_xA & 0 & 0 & 0 \\ 0 & v_xB & 0 & -v_xA \\ u_xB & 0 & 0 & 0 \end{pmatrix}.$$

By reading off the coefficients of y_1 (the [2, 1]-position) and x_0 (the [4, 1]-position), we obtain the system:

$$A_x - u_xA + v_xB = 0 \tag{6.14}$$

$$B_x - v_xA - u_xB = 0 \tag{6.15}$$

Now consider A (6.14) + B (6.15),

$$AA_x + BB_x - u_x(A^2 + B^2) = 0,$$

from which

$$\left(\log(A^2 + B^2)\right)_x = 2u_x,$$

so that

$$A^2 + B^2 = c_1^2 e^{2u}. \tag{6.16}$$

On the other hand, B (6.14) $-$ A (6.15) gives

$$BA_x - AB_x + v_x(A^2 + B^2) = 0,$$

from which

$$\frac{\left(\frac{A}{B}\right)_x}{1 + \frac{A^2}{B^2}} = v_x,$$

whence

$$\left(\tan^{-1}\left(\frac{A}{B}\right)\right)_x = v_x,$$

so that

$$\frac{A}{B} = \tan(v + c_2). \tag{6.17}$$

Finally, eliminating A and B from (6.16) and (6.17), we obtain

$$A = c_1 e^u \sin(v + c_2), \quad B = c_1 e^u \cos(v + c_2).$$

The c_2 constant may be ignored, as it merely results in a trivial translational symmetry. Thus, rewriting c_1 as c, we obtain the specialized form of q:

$$q = R + S = \begin{pmatrix} u_x + ce^u \sin v & 0 & 0 & 0 \\ 0 & -ce^u \sin v & 0 & v_x + ce^u \cos v \\ 0 & 0 & -u_x & 0 \\ 0 & -v_x & 0 & 0 \end{pmatrix}. \tag{6.18}$$

Having established the specialized form of q, we now calculate an associated zero-curvature system. This is of the form

$$[\mathcal{L}_q, D_t + a(q) + \Lambda'] = 0,$$

where the degree k of Λ' is odd, as prescribed by Theorem 6.1.1. In order to keep the computations as manageable as possible, we shall take $k = 3$. The zero-curvature equation then decomposes into:

$$\begin{aligned}
[q, \Lambda'] + [\Lambda, a_2(q)^{\perp}] &= 0 \quad (a_2(q)^{\perp}) && (6.19)\\
D_x a_2(q)^{\parallel} + [q, a_2(q)^{\perp}]^{\parallel} &= 0 \quad (a_2(q)^{\parallel}) && (6.20)\\
D_x a_2(q)^{\perp} + [q, a_2(q)]^{\perp} + [\Lambda, a_1(q)^{\perp}] &= 0 \quad (a_1(q)^{\perp}) && (6.21)\\
D_x a_1(q)^{\parallel} + [q, a_1(q)^{\perp}]^{\parallel} &= 0 \quad (a_1(q)^{\parallel}) && (6.22)\\
D_x a_1(q)^{\perp} + [q, a_1(q)]^{\perp} + [\Lambda, a_0(q)^{\perp}] &= 0 \quad (a_0(q)^{\perp}) && (6.23)\\
D_x a_0(q)^{\parallel} + [q, a_0(q)^{\perp}]^{\parallel} &= 0 \quad (a_0(q)^{\parallel}) && (6.24)\\
D_x a_0(q) + [q, a_0(q)] &= D_t q \quad (\mathrm{PDE}) && (6.25)
\end{aligned}$$

At the end of each equation is listed the entity that particular equation determines. The last equation produces the evolutionary system.

The Heisenberg element of degree 3 in this case is

$$\Lambda' = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -3\,\lambda \end{pmatrix}.$$

As $\mathcal{H}_2[w]$ is spanned by $\lambda y_1 + \lambda y_2 + [x_1, x_2]$ and $\mathcal{H}_2^{\perp}[w]$ by

$$\lambda y_1 - [x_1, x_2], \lambda y_2 - [x_1, x_2], \lambda x_0, \lambda x_3,$$

a typical element $a_2(q)$ of degree 2 in the $\mathsf{s}[w]$-gradation is then:

$$a_2(q) = \begin{pmatrix} 0 & 0 & \alpha_2 + \beta_2 + \gamma_2 & 0 \\ \lambda(\alpha_2 - \beta_2) & 0 & 0 & 0 \\ 0 & \lambda(\alpha_2 - \gamma_2) & 0 & \lambda\epsilon_2 \\ \lambda\delta_2 & 0 & 0 & 0 \end{pmatrix}$$

Here α_2 determines the $\mathcal{H}[w]$-component and the remaining entries the $\mathcal{H}^{\perp}[w]$-components. With the help of MAPLE throughout, we now begin solving Equations (6.19) through (6.24).

From (6.19):

$$\begin{aligned} \beta_2 &= 0 \\ \gamma_2 &= 0 \\ \delta_2 &= -4v_x \\ \epsilon_2 &= 4v_x + 4ce^u \cos v \end{aligned}$$

From (6.20),

$$\alpha_{2_x} = 0,$$

so that $\alpha_2 = 0$, the usual homogeneity requirements of differential degree applying.

Next, as $\mathcal{H}_1[w]$ is spanned by $\Lambda = x_1 + x_2 + \lambda[y_2, y_1]$ and $\mathcal{H}_1^{\perp}[w]$ by

$$\lambda[y_2, y_1] - x_1, \lambda[y_2, y_1] - x_2, y_3, y_0,$$

a typical element $a_1(q)$ of degree 1 is:

$$a_1(q) = \begin{pmatrix} 0 & \alpha_1 - \beta_1 & 0 & \epsilon_1 \\ 0 & 0 & \alpha_1 - \gamma_1 & 0 \\ \lambda(\alpha_1 + \beta_1 + \gamma_1) & 0 & 0 & 0 \\ 0 & 0 & \delta_1 & 0 \end{pmatrix}$$

Again, α_1 determines the $\mathcal{H}[w]$-component and the remaining entries determine the $\mathcal{H}^{\perp}[w]$-components. From (6.21):

$$\begin{aligned} \beta_1 &= \gamma_1 = \tfrac{4}{3}v_x(v_x + ce^u \cos v) \\ \delta_1 &= \epsilon_1 = -4v_x x + 4cv_x e^u \sin v + 4u_x v_x \end{aligned}$$

From (6.22),

$$\alpha_{1_x} = \tfrac{4}{3}ce^u(-v_x^2 \sin v + u_x v_x \cos v + v_{xx} \cos v) + \tfrac{8}{3}v_x v_{xx},$$

whence

$$\alpha_1 = 4(v_x^2 + cv_x e^u \cos v).$$

Finally, as $\mathcal{H}_0[w]$ is spanned by $h_1 + 2h_2 + 3h_3$ and $\mathcal{H}_0^{\perp}[w]$ by

$$h_1, h_2, [x_2, x_3], [y_3, y_2],$$

a typical element $a_0(q)$ of degree 0 is:

$$a_0(q) = \begin{pmatrix} \alpha_0 + \beta_0 & 0 & 0 & \\ 0 & \alpha_0 - \beta_0 + \gamma_0 & 0 & \delta_0 \\ 0 & 0 & \alpha_0 - \gamma_0 & 0 \\ 0 & \epsilon_0 & 0 & -3\alpha_0 \end{pmatrix}$$

Once again, α_0 determines the $\mathcal{H}[w]$-component and the remaining entries determine the $\mathcal{H}^{\perp}[w]$-components. From (6.23):

$$\begin{aligned} \beta_0 = \; & -4v_x v_{xx} + 4u_x v_x^2 + \tfrac{4}{3}ce^u(3v_x^2 \sin v - v_{xx} \cos v + u_x v_x \cos v) \\ & + \tfrac{4}{3}c^2 e^{2u} v_x \sin v \cos v \end{aligned}$$

$$
\begin{aligned}
\gamma_0 &= -4v_xv_{xx} + 4u_xv_x^2 + \tfrac{4}{3}ce^u(3v_x^2\sin v - 2v_{xx}\cos v + 2u_xv_x\cos v) \\
&\quad + \tfrac{8}{3}c^2e^{2u}v_x\sin v\cos v \\
\delta_0 &= 4v_{xxx} - 4u_{xx}v_x - 4u_x^2v_x - 4ce^u(3u_xv_x\sin v + v_x^2\cos v) - 4c^2e^{2u}v_x\sin^2 v \\
\epsilon_0 &= -4v_{xxx} + 4u_{xx}v_x + 4u_x^2v_x + 4ce^u(v_{xx}\sin v + 2u_xv_x\sin v + v_x^2\cos v)
\end{aligned}
$$

From (6.24),

$$
\begin{aligned}
\alpha_{0_x} &= \tfrac{4}{3}ce^u(u_xv_{xx}\sin v - v_xv_{xx}\sin v + v_{xxx}\cos v - u_{xx}v_x\cos v - u_x^2v_x\cos v) \\
&\quad + \tfrac{4}{3}c^2e^{2u}(v_x^2\sin^2 v - v_{xx}\sin v\cos v - 2u_xv_x\sin v\cos v),
\end{aligned}
$$

so that

$$
\alpha_0 = \tfrac{4}{3}ce^u(v_{xx} - u_xv_x)\cos v - \tfrac{4}{3}c^2e^{2u}v_x\sin v\cos v.
$$

Thus, the evolutionary system given by (6.25) is:

$$
\begin{aligned}
\tfrac{1}{4}u_t &= -v_xv_{xx} + u_xv_x^2 + ce^u((v_xu_x - v_{xx})\cos v + v_x^2\sin v) \\
&\quad + c^2v_xe^{2u}\sin v\cos v \\
\tfrac{1}{4}v_t &= v_{xxx} - u_{xx}v_x - u_x^2v_x - ce^u(3u_xv_x\sin v + v_x^2\cos v) \\
&\quad - c^2v_xe^{2u}\sin^2 v
\end{aligned}
$$

The factors of $\frac{1}{4}$ may be removed by rescaling the temporal variable t. Setting $c = 0$, so that $X = 0$, amounts to the specialized system corresponding to

$$
\tilde{\tau}(q) = q,
$$

over the Lie algebra of fixed points of $\tilde{\tau}$. This may be described as the following subalgebra of the loop algebra over $\mathfrak{a}_3$, with matrix representation $\mathfrak{sl}(4)$, namely the vector space direct sum of

$$
\begin{aligned}
A: &= \mathrm{Sp}\,\{\lambda^{2m} \otimes \{h_1 + h_2, x_1 + x_2, y_1 + y_2\} : m \in \mathbb{Z}\}, \\
B: &= \mathrm{Sp}\,\{\lambda^{2m+1} \otimes \{h_1 - h_2, x_1 - x_2, y_1 - y_2\} : m \in \mathbb{Z}\}, \\
C: &= \mathrm{Sp}\,\{\lambda^{2m} \otimes \{x_0 + x_3, y_0 + y_3, [x_2, x_3] - [y_3, y_2]\} : m \in \mathbb{Z}\}, \\
D: &= \mathrm{Sp}\,\{\lambda^{2m+1} \otimes \{h_1 + 2h_2 + 3h_3, x_0 - x_3, y_0 - y_3, [x_2, x_3] + [y_3, y_2]\} : m \in \mathbb{Z}\}.
\end{aligned}
$$

Note that A is a subalgebra and $A \oplus B \cong \mathfrak{a}_2^{(2)}$, as follows from [28, 25], since on the regular subalgebra $\mathfrak{a}_3^{(1)}$, $\tilde{\tau}$ acts as the extended Dynkin diagram symmetry, of which there is only one, up to isomorphism, for $\mathfrak{a}_n^{(1)}$, as observed in §5.1. Furthermore,

$$[C, C] \subset A, \quad [D, D] \subset A \oplus C, \quad [C, D] \subset B \oplus D.$$

Thus, we have a specialized zero-curvature system over a nonstandard twisted type loop algebra, which is a "deformation", in the sense of [26, 27, 39, 40] of the associated system with $c = 0$. Consequently, this specialization approach opens up a new class of integrable systems, previously unknown.

6.2.3 The Conjugacy Class of $r_{\alpha_1} r_{\alpha_3}$ in $\mathfrak{a}_3$

The extended Carter diagram in this case is given in Figure 6.3 and the effect of $\tilde{\tau}$ on the regular subalgebra indicated by the dotted axis of reflection. As an instance

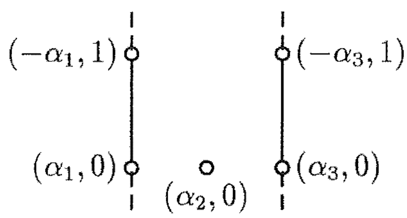

Figure 6.3: Extended Carter diagram for $w = r_{\alpha_1} r_{\alpha_3}$

of Case I of regular type, the order of w is preserved on lifting. From Theorem 5.3.4, we know that

$$\mathrm{s}[w] = (1, 1, -1, 1),$$

in accordance with Example 3.4.2. Moreover, Theorem 5.3.4, as once again verified by Example 3.4.2, shows that $\mathcal{H}[w]$ is generated by:

$$\lambda^n (h_1 + 2h_2 + h_3) \text{ with degree } 2n,$$

and

$$\lambda^n (\lambda y_1 + x_1) \text{ and } \lambda^n (\lambda y_3 + x_3) \text{ with degree } 2n + 1.$$

The regular elements are those of the form

$$\lambda^n(\lambda y_1 + x_1 + k(\lambda y_3 + x_3)), \quad k \neq 0, \pm 1.$$

Thus, the typical regular Heisenberg element of degree 1 is:

$$\Lambda = \lambda y_1 + x_1 + k(\lambda y_3 + x_3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & k\lambda & 0 \end{pmatrix}.$$

From §5.2, the action of $\tilde{w}$ on the simple roots of $\mathfrak{a}_3^{(1)}$ is:

$$\begin{aligned} (\alpha_1, 0) &\mapsto (-\alpha_1, 1) \\ (\alpha_2, 0) &\mapsto (\alpha_1 + \alpha_2 + \alpha_3, -1) \\ (\alpha_3, 0) &\mapsto (-\alpha_3, 1) \\ (\alpha_0, 1) &\mapsto (-\alpha_2, 0) \end{aligned}$$

Likewise, the action of $\tilde{\tau}$ is:

$$\begin{aligned} (\alpha_1, 0) &\mapsto (\alpha_1, 0) \\ (\alpha_2, 0) &\mapsto (-\alpha_1 - \alpha_2 - \alpha_3, 0) \\ (\alpha_3, 0) &\mapsto (\alpha_3, 0) \\ (\alpha_0, 1) &\mapsto (\alpha_2, 1) \end{aligned}$$

In accordance with Proposition 5.3.1, $\mathcal{H}_0^{\perp}[w]$ is spanned by

$$\{h_1, h_3, [x_1, x_2], [x_2, x_3], [y_2, y_1], [y_3, y_2]\}.$$

Now,

$$\tilde{\tau}[x_1, x_2] = [\tilde{\tau}(x_1), \tilde{\tau}(x_2)] = [x_1, x_0] = -[y_3, y_2],$$

and

$$\tilde{\tau}[x_2, x_3] = [\tilde{\tau}(x_2), \tilde{\tau}(x_3)] = [x_0, x_3] = -[y_2, y_1],$$

so that the subspace of $\mathcal{H}_0^{\perp}[w]$ of fixed points of $\tilde{\tau}$ is spanned by

$$\{h_1, h_3, [x_1, x_2] - [y_3, y_2], [x_2, x_3] - [y_2, y_1]\}.$$

Hence,

$$\begin{aligned} R &= r_x = u_x h_1 + v_x h_3 + w_x([x_1, x_2] - [y_3, y_2]) + z_x([x_2, x_3] - [y_2, y_1]) \\ &= \begin{pmatrix} u_x & 0 & w_x & 0 \\ 0 & -u_x & 0 & z_x \\ -z_x & 0 & v_x & 0 \\ 0 & -w_x & 0 & -v_x \end{pmatrix}, \end{aligned}$$

where u, v, w, z are functions of x and t. On the other hand, by Proposition 5.3.1, $\mathcal{H}_{-1}^{\perp}[w]$ is spanned by

$$\{\lambda^{-1}x_1 - y_1, \lambda^{-1}x_3 - y_3, x_2, \lambda^{-1}y_2, x_0, \lambda^{-1}y_0\}$$

and the subspace of eigenvectors of $\tilde{\tau}$ with eigenvalue -1 is spanned by

$$\{x_2 - x_0, \lambda^{-1}y_2 - \lambda^{-1}y_0\}.$$

Thus,

$$X = A(x_2 - x_0) + B(\lambda^{-1}y_2 - \lambda^{-1}y_0) = \begin{pmatrix} 0 & 0 & 0 & -\lambda^{-1}B \\ 0 & 0 & A & 0 \\ 0 & \lambda^{-1}B & 0 & 0 \\ -A & 0 & 0 & 0 \end{pmatrix}.$$

Now we are in a position to solve the specialization equation (6.13) of Corollary 6.1.5. First, we calculate $S = (\Lambda X)'^{\perp}$:

$$\Lambda X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & k\lambda & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -\lambda^{-1}B \\ 0 & 0 & A & 0 \\ 0 & \lambda^{-1}B & 0 & 0 \\ -A & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & A & 0 \\ 0 & 0 & 0 & -B \\ -kA & 0 & 0 & 0 \\ 0 & kB & 0 & 0 \end{pmatrix}$$

As the last matrix is trace free and has no $\mathcal{H}_0[w]$-component,

$$S = \begin{pmatrix} 0 & 0 & A & 0 \\ 0 & 0 & 0 & -B \\ -kA & 0 & 0 & 0 \\ 0 & kB & 0 & 0 \end{pmatrix}.$$

The specialization equation is

$$D_x X + [r_x, X] = 0,$$

of which the left hand side is

$$\begin{pmatrix} 0 & 0 & 0 & -\lambda^{-1}B_x \\ 0 & 0 & A_x & 0 \\ 0 & \lambda^{-1}B_x & 0 & 0 \\ -A_x & 0 & 0 & 0 \end{pmatrix}$$
$$+ \left[\begin{pmatrix} u_x & 0 & w_x & 0 \\ 0 & -u_x & 0 & z_x \\ -z_x & 0 & v_x & 0 \\ 0 & -w_x & 0 & -v_x \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -\lambda^{-1}B \\ 0 & 0 & A & 0 \\ 0 & \lambda^{-1}B & 0 & 0 \\ -A & 0 & 0 & 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & 0 & 0 & -\lambda^{-1}B_x \\ 0 & 0 & A_x & 0 \\ 0 & \lambda^{-1}B_x & 0 & 0 \\ -A_x & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \lambda^{-1}w_xB & 0 & -\lambda^{-1}u_xB \\ -z_xA & 0 & -u_xA & 0 \\ 0 & \lambda^{-1}v_xA & 0 & \lambda^{-1}z_xB \\ v_xA & 0 & -w_xA & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & \lambda^{-1}w_xB & 0 & \lambda^{-1}v_xB \\ -z_xA & 0 & v_xA & 0 \\ 0 & -\lambda^{-1}u_xA & 0 & \lambda^{-1}z_xB \\ -u_xA & 0 & -w_xA & 0 \end{pmatrix}$$

By reading off the coefficients of x_2 (the $[2,3]$-position) and y_2 (the $[3,2]$-position), we obtain the system:

$$A_x - (u_x + v_x)A = 0 \tag{6.26}$$

$$B_x + (u_x + v_x)B = 0 \tag{6.27}$$

These equations lead to

$$A = c_1 e^{u+v}, \quad B = c_2 e^{-(u+v)}.$$

Thus, the specialized form of q is:

$$q = R + S = \begin{pmatrix} u_x & 0 & w_x + c_1e^{u+v} & 0 \\ 0 & -u_x & 0 & z_x - c_2e^{-(u+v)} \\ -z_x - kc_1e^{u+v} & 0 & v_x & 0 \\ 0 & -w_x + kc_2e^{-(u+v)} & 0 & -v_x \end{pmatrix}. \tag{6.28}$$

The associated evolutionary system, even for Λ' of degree 3, appears to be of little interest and so is not reproduced here.

6.3 Concluding Remarks

The research contained in this thesis generalizes the work of Guil [25] by setting it within the framework of the theory of generalized Drinfel'd-Sokolov hierarchies as developed by de Groot *et al.* [13, 7]. In doing so, it has established the existence of a new class of integrable systems by showing that they are associated with nonstandard twisted type algebras given by the fixed point subspace of the automorphism $\tilde{\tau}$ (or $\tilde{\tau}'$) constructed in Chapter 5.

Despite the fact that only algebras of type $\mathfrak{a}$ have been explored here, the construction of the automorphisms $\tilde{\tau}$ and $\tilde{\tau}'$ is sufficiently general to allow the possibility of extending it to the other algebras. Given the Weyl group element, the root space automorphism τ is defined to be the extension of a diagram symmetry on each of the simple factors of the corresponding regular subalgebra and to act as $-\mathrm{id}$ on the fixed points of w. Hopefully, this will prove to be a root space automorphism as it did for the case of $\mathfrak{a}_n$ in Theorem 5.1.1. Some sort of general argument involving the fundamental dominant weights would be necessary here.

Unfortunately, no satisfactory theory could be developed for the case when w doubles its order upon lifting. This is a matter worthy of future attention.

There is also the possibility of extending this theory to partially modified generalized Drinfel'd-Sokolov hierarchies associated with a gradation vector $\mathbf{s}$, where $\mathbf{s} \prec \mathbf{s}[w]$. This would involve issues of gauge fixing as mentioned in §4.1.

Finally, there may be some sort of specialization theory for hierarchies of Type II, *i.e.* when w does *not* admit a regular eigenvector. In the case of GD-S hierarchies alone, there is still scant knowledge of these, though some forays have been made by Fordy [22] and, from a different point of view, by McIntosh [43].

Appendix A

Another Specialization for the Coxeter Class

In [25], Guil constructs the component S of the specialized form of $q = R+S$ of Corollary 6.1.5, by showing that S lies in the fixed point subspace in $\mathcal{H}_0^{\perp}[w_C] = \mathcal{H}_0[w_C]$ of the automorphism $\tilde{\tau}\tilde{w}_C$. This approach, as already observed in the Remark of §6.2.1, does not lend itself to obvious generalization for arbitrary w. However, it may be used to facilitate the calculation when $w = w_C$ of a specialized hierarchy obtained from setting $X_3 = 0$ in (6.2), so that

$$K = I + X_1(q, \tilde{\tau}(q)) + X_2(q, \tilde{\tau}(q)).$$

Instead of defining the second automorphism φ to be $\tilde{\tau}\tilde{w}_C$, as in [25][1], to obtain a direct sum decomposition of $\mathcal{H}_0[w_C]$ into the fixed points of $\tilde{\tau}$ and φ, it is more convenient in this case to define

$$\varphi := \tilde{\tau}\tilde{w}_C^2.$$

When n is *odd*, the fixed point subspaces of $\tilde{\tau}$ and φ are complementary on restriction to $\mathcal{H}[w_C]$.

[1]The automorphism φ here corresponds to Guil's automorphism ν. However, ν has already been used in this thesis to denote a Dynkin diagram symmetry of $\mathfrak{g}$., hence the change of notation.

REMARK: This is not the case when n is even, since then $\tilde{\tau}$ and φ have nonzero fixed points in common. Even if $\tilde{\tau}$ is replaced by the automorphism $\tilde{\tau}' = \tilde{\tau}\tilde{w}_C$ of Theorem 5.3.7, the join of the two fixed point subspaces is not all of $\mathcal{H}_0[w_C]$, so that this construction will not produce the required direct sum decomposition of $\mathcal{H}_0[w_C]$ when n is even. □

It may be shown [25] that $\tilde{w}_C$ acts as conjugation by Λ^{-1}, so that

$$\tilde{w}_C^2(X) = \Lambda^{-2}X\Lambda^2,\ \forall X \in \mathfrak{a}_n^{(1)}.$$

Note that this is special to the Coxeter case.

The restrictions of the maps $\mathrm{id} - \tilde{w}_C^2$ and $\mathrm{id} - \tilde{w}_C^{-2}$ to $\mathcal{H}_0[w_C]$ are both bijective and therefore have inverses. In fact, restricted to $\mathcal{H}_0[w_C]$:

$$\begin{aligned} (\mathrm{id} - \tilde{w}_C^{-2})^{-1} &= \frac{1}{n}\sum_{k=1}^{n} k\tilde{w}_C^{2k} \\ (\mathrm{id} - \tilde{w}_C^{2})^{-1} &= \frac{1}{n}\sum_{k=1}^{n} k\tilde{w}_C^{-2k} \end{aligned} \qquad (A.1)$$

These formulae may be used to explicitly calculate the projections π_φ and π_τ onto the respective fixed point subspaces in $\mathcal{H}_0[w_C]$, where, after some manipulation of summations, it may be shown that

$$\begin{aligned} \pi_\varphi &= (\mathrm{id} - \tilde{w}_C^{2})^{-1}(\mathrm{id} - \tau) \\ \pi_\tau &= (\mathrm{id} - \tilde{w}_C^{-2})^{-1}(\mathrm{id} - \varphi), \end{aligned} \qquad (A.2)$$

where the maps are all understood to be restricted to $\mathcal{H}_0[w_C]$.

On stipulating that $X_3(q, \tilde{\tau}(q)) = 0$, the specialization equations of (4.8) and (4.9) become:

$$\tilde{\tau}(q) - q + [\Lambda, X_1] = 0 \qquad (A.3)$$

$$D_x X_1 + \tilde{\tau}(q)X_1 - X_1 q + [\Lambda, X_2] = 0 \qquad (A.4)$$

$$D_x X_2 + \tilde{\tau}(q)X_2 - X_2 q = 0 \qquad (A.5)$$

The same kind of argument used in the proof of Theorem 6.1.3 shows that X_1 takes values in $\mathcal{H}_{-1}^{\perp}[w_C]$ and that $\tilde{\tau}(X_1) = -X_1$.

An explicit calculation is presented for the algebra $\mathfrak{a}_3^{(1)}$ when $n = 3$. The action of φ on $\mathcal{H}_0[w_C]$ is:

$$\begin{aligned} h_1 &\mapsto h_0 = -h_1 - h_2 \\ h_2 &\mapsto h_2 \end{aligned}$$

Thus the fixed point subspace of φ when restricted to $\mathcal{H}[w_c]$ is spanned by h_2. The fixed point subspace of $\tilde{\tau}$ when restricted to $\mathcal{H}[w_C]$ is spanned by $h_1 + h_2$.

Let $q = R + S$ under the direct sum decomposition of $\mathcal{H}_0[w_C]$ into the restricted fixed point subspaces of $\tilde{\tau}$ and φ, so that $\tilde{\tau}(R) = R$ and $\varphi(S) = S$. Since $\tilde{\tau} = \tilde{w}_C^2\varphi$, (A.3) becomes

$$\tilde{w}_C^2(S) - S + [\Lambda, X_1] = 0, \tag{A.6}$$

where $X_1 = m(y_1 - y_2)$, for some function m, since X_1 takes values in $\mathcal{H}_{-1}^{\perp}[w_C]$ and $\tilde{\tau}X_1 = -X_1$. Applying (A.1) then leads to

$$S = \tfrac{1}{3}\sum_{k=1}^{3} k\tilde{w}_C^{-k}[\Lambda, X_1] = -mh_2.$$

Proceeding to (A.5), the definitions of R and S lead to

$$\tilde{\tau}(q)X_2 - X_2 q = [R, X_2] + \tilde{w}_C^2(S)X_2 - X_2 S.$$

Noting that X_2 takes values in $\mathcal{H}_{-2}[w_C]$ and the matrix Λ^{-2} corresponds to the representation of $x_0 + \lambda^{-1}x_1 + \lambda^{-1}x_2$, it follows that $X_2 = A\Lambda^{-2}$, where $A = \mathrm{diag}(n_0, n_1, n_2)$, so that $X_2 = n_0 x_0 + n_1\lambda^{-1}x_1 + n_2\lambda^{-1}x_2$. Therefore,

$$\begin{aligned} \tilde{w}_C^2(S)X_2 - X_2 S &= \tilde{w}_C^2(S)A\Lambda^{-2} - A\Lambda^{-2}S \\ &= A\tilde{w}_C^2(S)\Lambda^{-2} - A\Lambda^{-2}S \\ &= A(\Lambda^{-2}S\Lambda^2)\Lambda^{-2} - A\Lambda^{-2}S = 0. \end{aligned}$$

In the last line, use was made of the fact that $\tilde{w}_C^2(W) = \Lambda^{-2}W\Lambda^2$. Note also that $\tilde{w}_C^2(S)$ and A commute as both are elements of $\mathcal{H}_0[w_C] = \mathfrak{h}$, which is Abelian.

Equation (A.5) now becomes

$$D_x X_2 + [R, X_2] = 0.$$

The solution of this equation is neatly expressible in the form,

$$X_2 = c_0 \exp(\alpha_0(r))x_0 + c_1 \exp(\alpha_1(r))\lambda^{-1}x_1 + c_2 \exp(\alpha_2(r))\lambda^{-1}x_2,$$

for arbitrary constants c_0, c_1, c_2 where

$$R = r_x = p_x(h_1 + h_2),$$

for some function p. Since $\alpha_i(h_j) = a_{ij}$, it follows that

$$n_0 = c_0 e^{2p},\ n_1 = c_1 e^{-p},\ n_2 = c_2 e^{-p}.$$

Finally, substitution of these results into (A.4) leads to the following system:

$$\begin{aligned} (c_2 - c_1)e^{-p} &= 0 \\ m_x - p_x m + m^2 + c_0 e^{2p} - c_2 e^{-p} &= 0 \\ -m_x + p_x m - m^2 + c_1 e^{-p} - c_0 e^{2p} &= 0 \end{aligned}$$

Thus, $c_1 = c_2$ and letting $c_0 = -C^2$ and $p = \log u_x$, it follows that

$$m_x - \frac{u_{xx}}{u_x}m + m^2 = \frac{c_1}{u_x} + C^2 u_x^2. \tag{A.7}$$

In order to solve for m as a function of u and its derivatives, let m be of the form $m(u, u_x)$, so that now,

$$m_x = D_x m = \tfrac{\partial m}{\partial u}u_x + \tfrac{\partial m}{\partial u_x}u_{xx}.$$

On substitution into (A.7), the u_{xx} coefficient gives

$$\tfrac{\partial m}{\partial u_x}u_x - m = 0,$$

so that

$$m = g(u)u_x,$$

for some function g. Comparing coefficients of u_x yields $c_1 = 0$ and

$$g' + g^2 = C^2, \tag{A.8}$$

for which the standard linearization procedure produces the solution,

$$m = Cu_x \left(\frac{e^{Cu} - Ke^{-Cu}}{e^{Cu} + Ke^{-Cu}} \right), \tag{A.9}$$

with arbitrary constant, K.

Thus,

$$\begin{aligned}
q &= R + S \\
&= p_x(h_1 + h_2) - m_2 h_2 \\
&= \frac{u_{xx}}{u_x}(h_1 + h_2) - Cu_x \left(\frac{e^{Cu} - Ke^{-Cu}}{e^{Cu} + Ke^{-Cu}} \right) h_2 \\
&= \operatorname{diag} \left(\frac{u_{xx}}{u_x}, -Cu_x \left(\frac{e^{Cu} - Ke^{-Cu}}{e^{Cu} + Ke^{-Cu}} \right), -\frac{u_{xx}}{u_x} + Cu_x \left(\frac{e^{Cu} - Ke^{-Cu}}{e^{Cu} + Ke^{-Cu}} \right) \right), \\
X_1 &= Cu_x \left(\frac{e^{Cu} - Ke^{-Cu}}{e^{Cu} + Ke^{-Cu}} \right) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \\
X_2 &= -C^2 u_x^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

As an example of a hierarchy associated with this specialized q, take Λ' of degree 5, so that an evolution equation for u results from the zero-curvature condition

$$[D_x + q + \Lambda, D_t + a(q) + \Lambda'] = 0.$$

Alternatively, the Hamiltonian approach in [25, §5], [24] may be employed.[2] The outcome is:

$$-9u_t = u_{xxxxx} - 5u_x^{-1}u_{xx}u_{xxxx} + 5u_x^{-2}u_{xx}^2u_{xxx} + 5C^2u_xu_{xx}^2 - 5C^2u_x^2u_{xxx} + C^4u_x^5$$

[2]Note the incorrect Hamiltonian in Equation (5.4) of [25]. It should be: $H = \frac{1}{3}q_{1x}^2 - q_0q_{1x} + q_0^2 - \frac{1}{9}q_1^3$. The evolution equation for $X_2 = 0$ derived from this Hamiltonian is also incorrect.

Note that the equation is independent of K. So there is a 1-parameter family of zero-curvature representations for this equation, as q depends on K. The calculations here have been performed with the assistance of MAPLE.

For $C = 0$, solving Equation (A.8) yields

$$q = \operatorname{diag}\left(\frac{u_{xx}}{u_x}, -\frac{u_x}{u+K}, -\frac{u_{xx}}{u_x} + \frac{u_x}{u+K}\right),$$

which has corresponding equation the same as the above with $C = 0$. This may be related to the equation for the $X_2 = 0$ specialization, namely,

$$\begin{aligned} -9w_t &= w_{xxxxx} - 5w_x w_{xx}^2 - 5w_x^2 w_{xxx} + 5w_{xx}w_{xxx} + w_x^5 \\ &\quad -5c^2 w_{xxx}e^{2w} - 15c^2 w_x w_{xx} e^{2w} + 5c^4 w_x e^{4w}, \end{aligned}$$

by the change of variables,

$$ce^w = \frac{u_x}{u+K}.$$

Furthermore, letting $K \to \infty$ gives back the specialization for $X_1 = 0$ with the change of variables, $s = u_{xx}/u_x$. The equation in that case is:

$$-9s_t = s_{xxxxx} - 5s^2 s_{xxx} + 5s_x s_{xxx} + 5s_{xx}^2 - 5s_x^3 - 20ss_x s_{xx} + 5s^4 s_x$$

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